A Mathematics Primer for Physics Graduate Students (Version 2.0)

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A Note of Explanation

It has been my experience that many incoming graduate students have not yet been exposed to some of the more advanced mathematics needed to access electrodynamics or quantum mechanics. Mathematical tools such as tensors or di erential forms, intended to make the calculations much simpler, become hurdles that the students must overcome, taking away the time and understanding from the physics. The professors, on the other hand, do not feel it necessary to go over these topics. They cannot be blamed, since they are trying to teach physics and do not want to waste time with the details of the mathematical trickery that students should have picked up at some point during their undergraduate years. Nonetheless, students nd themselves at a loss, and the professors often don't even know why the students are struggling until it is too late.

Hence this paper. I have attempted to summarize most of the more advanced mathematical trickery seen in electrodynamics and quantum mechanics in simple and friendly terms with examples. I do not know how well I have succeeded, but hopefully students will be able to get somerelief from this summary.

Please realize that while I have done my best to make this as complete as possible, it is not meant as a textbook. I have left out many details, and in some cases I have purposely glossed over vital subtlties in the de nitions and theorems. I do not provide many proofs, and the proofs that are here are sketchy at best. This is because I intended this paper only as a reference and a source. I hope that, in addition to giving students a place to look for a quick reference, it might spark some interest in a few to take a math course or read a book to get more details. I have provided a list of references that I found useful when writing this review at the end.

Finally, I would like to thank Brock Tweedie for a careful reading of the paper, giving very helpful advice on how to improve the language as well as catching a few mistakes.

Good luck, and have fun!

September 19, 2006

In this version (1.5), I have made several corrections and improvements. I would like to write a chapter on analysis and conformal mapping, but unfortunately I haven't got the time right now, so it will have to wait. I want to thank Linda Carpenter and Nadir Jeevanjee for

List of Symbols and Abbreviations

Here is a list of symbols used in this review, and often by professors on the blackboard.

Symbol	Meaning
8	For all
9	There exists
9!	There exists unique
a 2 A	a is a member (or element) of the set A
A B	A is a subset of B
A = B	A B and B A
A [B	The set of members of the sets A or B
A \ B	The set of members of the sets A and B
;	The empty set; the set with no elements
A q B	The \disjoint union"; same as A [B where A \ B = ;
WLOG	Without Loss of Generality
p) q	implies; If p is true, then q is true.
p , q; i	p is true if and only if q is true.
N	Set of natural numbers (positive integers)
Z	Set of integers
Q	Set of rational numbers
R	Set of real numbers
C	Set of complex numbers

Contents

Chapter 1

Tensors and Such

1.1 Some Definitions

Before diving into the details of tensors, let us review how coordinates work. In N-dimensions, we can write N unit vectors, defining a basis. If they have unit magnitude and point in orthogonal directions, it is called an orthonormal basis, but this does not have to be the case. It is common mathematical notation to call these vectors $\hat{\mathbf{e}}_i$, where i is an index and goes from 1 to N. It is important to remember that these objects are vectors even though they also have an index. In three dimensions, for example, they may look like:

$$\hat{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \hat{\mathbf{e}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \hat{\mathbf{e}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.1.1}$$

Now you can write a general N-vector as the sum of the unit vectors:

$$\vec{r} = x^1\hat{\mathbf{e}}_1 + x^2\hat{\mathbf{e}}_2 + \cdots + x^N\hat{\mathbf{e}}_N = \sum_{i=1}^{N} x^i\hat{\mathbf{e}}_i \tag{1.1.2}$$

where we have written x^i as the coordinates of the vector $\vec{r}$. Note that the index for the coordinate is a superscript, whereas the index for the unit vector is a subscript. This small fact will turn out to be very important! Be careful not to confuse the indices for exponents.

Before going any further, let us introduce one of the most important notions of mathematics: the inner product. You are probably very familiar with the concept of an inner product from regular linear algebra. Geometrically, we can define the inner product in the following way:

$$\langle \vec{r}_1, \vec{r}_2 \rangle = \left\langle \sum_{i=1}^{N} x_1^i \hat{\mathbf{e}}_i, \sum_{j=1}^{N} x_2^j \hat{\mathbf{e}}_j \right\rangle = \sum_{i=1}^{N}\sum_{j=1}^{N} x_1^i x_2^j \langle \hat{\mathbf{e}}_i, \hat{\mathbf{e}}_j \rangle = \sum_{i=1}^{N}\sum_{j=1}^{N} x_1^i x_2^j g_{ij} \tag{1.1.3}$$

where we have defined the new quantity:

$$g_{ij} \equiv \langle \hat{\mathbf{e}}_i, \hat{\mathbf{e}}_j \rangle \tag{1.1.4}$$

Mathematicians sometimes call this quantity the first fundamental form. Physicists often simply call it the metric.

Now look at the last equality of Equation 1.1.3. Notice that there is no explicit mention of the basis vectors $\{\mathbf{e}_i\}$. This implies that it is usually enough to express a vector in terms of its components. This is exactly analogous to what you are used to doing when you say, for example, that a vector is (1,2) as opposed to $x+2y$. Therefore, from now on, unless it is important to keep track of the basis, we will omit the basis vectors and denote the vector only by its components. So, for example, $\mathbf{r}$ becomes x^i. Note that \

that the indices should be read as \ij " and not as \ji ". This can prove to be very important, as in general, the matrices we consider are not symmetric, and it is important to know the order of indices. Finally, even though we have only calculated this to lowest order, it turns out that Equation 1.2.8 has a generalization to large transformations. We will discuss these details in Chapter 2.

Let us consider a simple example. Consider a 2-vector, and let the transformation be a regular rotation by an angle . We know how to do such transformations. The new coordinates are:

$$\begin{aligned} x^{1'} &= x^1 \cos + x^2 \sin \quad x^1 + x^2 + O(^2) \\ x^{2'} &= x^2 \cos \quad x^1 \sin \quad x^2 \quad x^1 + O(^2) \end{aligned}$$

Now it is a simple matter of reading o the $A^{i'}_{:j}$:

$$\begin{aligned} A^{1'}_{:1} = 0 \quad A^{1'}_{:2} = \\ A^{2'}_{:1} = \quad A^{2'}_{:2} = 0 \end{aligned}$$

Armed with this knowedge, we can now de ne a tensor:

A tensor is a collection of objects which, combined the right way, transform the same way as the coordinates under in nitesimal (proper) symmetry transformations.

What are \in nitesimal proper symmetry transformations"? This is an issue that we will tackle in Chapter 2. For now it will be enough to say that in N-dimensions, they are translations and rotations in each direction; without the proper, they also include re ections ($x \; ! \quad x$). This de nition basically states that a tensor is an object that for each index, transforms according to Equation 1.2.8, with the coordinate replaced by the tensor.

Tensors carry indices. The number of indices is known as the tensor rank . Notice, however, that not everything that caries an index is a tensor!!

Some people with more mathematical background might be upset with this de nition. However, you should be warned that the mathematician's de nition of a tensor and the physicist's de nition of a tensor are not exactly the same. I will say more on this in Chapter 5. For the more interested readers, I encourage you to think about this; but for the magg40h3r(it)27(y)]TJ 0 -

of the origin, for example). So we will de ne a vector with a lower index as an object which leaves the length invariant:

$$\sum_{i=1}^{N} x_{i'} x^{i'} = s^2 = \sum_{i'=1}^{N} \left(\sum_{k=1}^{N} A^{i'}_{:k} x^k \right) \left(\sum_{l=1}^{N} x_l B^{l}_{:i'} \right) = \sum_{k,l=1}^{N} \left[\sum_{i'=1}^{N} B^{l}_{:i'} A^{i'}_{:k} \right] x^k x_l \tag{1.3.11}$$

where $B^{l}_{:i'}$ is the matrix that de nes how this new type of vector is supposed to transform. Notice that if the left and right side of this expression are supposed to be equal, we have a

Figure 1.1: A graph showing the difference between covariant and contravariant coordinates.

$$x_i = \sum_{j=1}^{N} g_{ij}\, x^j \tag{1.3.16}$$

$$x^j = \sum_{k=1}^{N} g^{jk}\, x_k \tag{1.3.17}$$

$$\sum_{j=1}^{N} g_{ij}\, g^{jk} = \delta_i^k \tag{1.3.18}$$

So we have found that the metric takes contravariant vectors to covariant vectors and vice versa, and in doing so we have found that the inverse of the metric with lower indices is the metric with upper indices. This is a very important identity, and will be used many times.

When working in an orthonormal flat-space coordinate system (so $g_{ij} = \delta_{ij}$), the difference between covariant and contravariant is negligible. We will see in Chapters 3 and 4 some examples of where the difference begins to appear. But even now I can show you a simple nontrivial example. Consider a cartesian-like coordinate system, but now let the coordinate axes cross at an angle of 60 deg (see Figure 1.1). Now the metric is no longer proportional to the Kroneker delta (compare this to Equation (1.1.4), and so there will be a difference between covariant and contravariant coordinates. You can see that explicitly in the figure above, where the same point has $x^1 < x^2$, but $x_1 > x_2$. So we find that covariant and contravariant are the same only in orthonormal cartesian coordinates! In the more general case, even in flat space, the difference is important.

As another example, consider polar coordinates $(r; \theta)$. Even though polar coordinates describe the flat plane, they are not orthonormal coordinates. Specifically, if $x^i = (r; \theta)$, then it is true that $x_i = (r; r^2\theta)$, so here is another example of how covariant and contravariant makes a difference.

1.4 Mixed Tensors

We have talked about covariant and contravariant tensors. However, by now it should be clear that we need not stop there. We can construct a tensor that is covariant in one index and contravariant in another. Such a tensor would be expected to transform as follows:

$$T^{i'}_{:j'} = \sum_{k,l=1}^{N} A^{i'}_{:k} A^{l}_{:j'} T^{k}_{:l} \tag{1.4.19}$$

Such objects are known as mixed tensors . By now you should be able to construct a tensor of any character with as many indices as you wish. Generally, a tensor with n upper indices and l lower indices is called a (n; l) tensor.

1.5 Einstein Notation

The last thing to notice from all the equations so far is that every time we have an index that is repeated as a superscript and a subscript, we are summing over that index. This is not a coincidence. In fact, it is generally true that anytime two indices repeat in a monomial, they are summed over. So let us introduce a new convention:

Anytime an index appears twice (once upstairs, once downstairs) it is to be summed over, unless speci cally stated otherwise.

This is known as Einstein notation , and will prove very useful in keeping the expressions simple. It will also provide a quick check: if you get an index repeating, but both subscripts or superscripts, chances are you made a mistake somewhere.

1.5.1 Another Shortcut

There is another very useful shortcut when writing down derivatives:

$$f(x)_{;i} = \partial_i f(x) \quad \frac{\partial f}{\partial x^i} \tag{1.5.20}$$

Notice that the derivative with respect to contravariant-x is covariant; the proof of this follows from the chain rule and Equation (1.2.6) This is very important, and is made much more explicit in this notation. I will also often use the shortand:

$$\partial^2 = \partial_i \partial^i = r^{-2} \tag{1.5.21}$$

This is just the Laplacian in arbitrary dimensions[1]. It is not hard to show that it is a scalar operator. In other words, for any scalar (x), ∂^2 (x) is also a scalar.

[1] In Minkowski space, where most of this is relevant for physics, there is an extra minus sign, so $\partial^2 = = \frac{\partial^2}{\partial t^2} - r^{-2}$, which is just the D'Alembertian (wave equation) operator. I will cover this more in Chapter 3; in this chapter, we will stick to the usual Euclidean space.

1.6 Some Special Tensors

1.6.1 Kroneker Delta

We have already seen the Kroneker Delta in action, but it might be useful to quickly sum up its properties here. This object is a tensor which always carries two indices. It is equal to +1 whenever the two indices are equal, and it is 0 whenever they are not. Notice that in

N=2: ε_{ij}

For the case of two dimensions, the Levi-Civita symbol has two indices, and we can write out very easily what it looks like:

Speci c permutations are usually written as a row of numbers. It is certainly true (and is rigorously provable for the anal retentive!) that any nite set of order n (i.e.: with n elements) can be represented by the set of natural numbers between 1 and n. Then WLOG[2] we can always represent our set S in this way. Then a permutation might look like this:

= (1234)

This permutation sends 1 7! 2,2 7! 3,3 7! 4,4 7! 1. This is in general how it always works.

What this tells us is that the set of even permutations is closed, and hense forms a subgroup, called the Alternating Subgroup (denoted A_n) of order $\frac{1}{2}n!$. Again, I omit details of the proof that this is indeed a subgroup, although it's pretty easy. Notice that the odd elements do not have this property. The alternating subgroup is a very powerful tool in mathematics.

For an application of permutations, a good place to look is index manipulation (hence why I put it in this chapter!). Sometimes, you might want to sum over all combinations of indices. For example, the general formula for the determinant of an n n matrix can be written as:

$$\det A = \sum_{2S_n} \operatorname{sgn}(\ )a_{1\ (1)}a_{2\ (2)}\quad a_{n\ (n)} \tag{1.7.27}$$

1.8 Constructing and Manipulating Tensors

To nish this chapter, I will review a few more manipulations that provide ways to construct tensors from other tensors.

1.8.1 Constructing Tensors

Consider two tensors of rank k (K) and l (L) respectively. We want to construct a new tensor (T) from these two tensors. There are three immediate ways to do it:

1. Inner Product : We can contract one of the indices from K and one from L to form a tensor T of rank k + l 2. For example, if K is rank 2 and L is rank 3, then we can contract, say, the rst index of both of them to form T with rank 3:

   $$T \quad = K_{:}\ L$$

   We could construct other tensors as well if we wished by contracting other indices. We could also use the metric tensor or the Levi-Civita tensor if we wished to form symmetric or antisymmetric combinations of tensors. For example, if A; B are vectors in 4 dimensions:

   $$T \quad = \qquad A\ B\ ) \quad T_{12} = A^3B^4 \quad A^4B^3; \quad \text{etc:}$$

2. Outer Product : We can alwayscan1.9552 a02997017-2(can1.9552 a0213(s5D]TJ/F2952 Tf 10.s3(s5DT

$$A_{(i}B_{j)} \qquad \frac{1}{2}(A_iB_j + A_jB_i) \tag{1.8.28}$$

$$A_{[i}B_{j]} \qquad \frac{1}{2}(A_iB_j - A_jB_i) \tag{1.8.29}$$

$$T = C \quad D \quad E = \begin{pmatrix} C & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & E \end{pmatrix} \tag{1.8.32}$$

where C is an m m matrix, D is an n n matrix, E is an l l matrix,and \0" is just lling in any left over entries with a zero. Notice that the nal matrix is an (m + n + l) (m + n + l) matrix.

At this point, we can rewrite our equation for the dyadic:

$$A \quad B \quad C \quad D \quad E \tag{1.8.33}$$

where A B is a 3 3 matrix, C is 1 1, D is 3 3 and E is 5 5. Notice that the matrix dimensions are not the same on both sides of the equation - these are not the same matrix! However, they each have the same number of independent quantities, and both represent the same tensor T. For this reason, the study of nding di erent dimensional matrices for T goes under the heading of representation theory . It is a very active part of modern algebraic research, and is very useful in many areas of science.

Why would one ever want to do this? The answer is that when working with dyadics in practice, very often one is only interested in the part of the tensor that transforms the same way. Notice that when you rotate T, you never mix terms between C, D and E. This kind of relationship is by no means as obvious when written in its Cartesian form. For that reason, it is often very useful to work with the separate spherical tensors rather than with the entire Cartesian tensor. The best example is in Quantum Mechanics, when you are talking about angular momentum. See Sakurai for more details.

Chapter 2

Transformations and Symmetries

2.1 Symmetries and Groups

Here we will discuss some properties of groups in general that you need to know. Then we will progress to discuss a very special and important kind of group, called a Lie Group. Finally we will look at some examples.

De nition: A group G is a set with an operator called multiplication that obeys four axioms: Closure ($g_1; g_2 \in G$) $g_1 g_2 \in G$), Associative($(g_1 g_2) g_3 = g_1 (g_2 g_3)$), an identity (generically called e such that $ge = eg = g$) and inverses (g has inverse g^{-1} such that $gg^{-1} = g^{-1} g = e$). If the group is commutative ($g_1 g_2 = g_2 g_1$), it is said to be an Abelian group , otherwise it is Non-Abelian.

Discrete groups represent operations that either do or do not take place. Continuous groups represent operations that di er in nitesimally from each other.

Elements in continuous groups can be labeled by a set of continuous parameters. If the set of parameters is nite, the group is said to be a nite group . If the ranges of the parameters are closed and bounded, the group is said to be a compact group .

De nition: A subgroup is a subset of a larger group that is also a group. A subgroup $N \quad G$ is a normal subgroup (also called invariant subgroup) if $8s_i \in N; g^{-1} s_i g \in N \; 8g \in G$. In other words, elements of N stay in N after being multiplied by elements in G. We write N / G. There are several de nitions for invariant subgroups, but they are all equivalent to this one.

Consider two groups that commute with each other $(G; H)$. Then you can construct the direct product group in the following way: $G \quad H \quad f (g_j h_k) j (g_j h_k)(g_l h_m) = (g_j g_l h_k h_m) g$. Be careful not to confuse a direct product group with a Kroneker product of tensors.

Fact: $G; H / G \quad H$ are invariant subgroups.

De nition: If a group has no invariant subgroups, it is called a simple group.

De nition: Suppose there is a square n n matrix M that corresponds to each element of a group G, where $g_1 g_2 = g_3$) $M(g_1)M(g_2) = M(g_3)$ and $M(g^{-1}) = [M(g)]^{-1}$. Then M is called a representation of the group G, and n is called the dimension of the representation. If this identi cation is one-to-one, the representation is called \faithful". An example of an unfaithful representation is where all elements of a group G go to 1.

De nition: A representation is reducible if each matrix can be expressed in block-diagonal form. Using the Kroneker sum notation, we can rephrase this de nition: 9A; B such that M = A B. A and B are also valid representations of the group. Furthermore, they do not talk to each other (think angular momentum in quantum mechanics). If a representation is not reducible, it is said to be irreducible .

Consider an n-tuple $_a$, not necessarely a vector in the sense of Chapter 1. Then

When is small, we can expand the exponential to get:

$$M(\ ) = I + i\ ^{j} F_j + O(\ ^2)$$

This is exactly the same as an in nitesimal transformation, and so we say that F_j generatesin nitesimal transformations.

Theorem: The local structure of any Lie group is completely speci ed by a set of commutation relations among the generators:

$$[F_i, F_j] = \sum_k c_{ijk} F_k$$

2.1.1 SU(2) - Spin

Consider a complex spinor

$$= \begin{matrix} 1 \\ 2 \end{matrix}$$

and the set of all transformations which leave the norm of the spinor invariant; i.e.:

$$s^2 = j\ ^1j^2 + j\ ^2j^2$$

s invariant under all transformations. We will also require det M = 1. From the above, we know that the generators (call them $_i$) must be linearly independent, traceless, Hermitian 2 2 matrices. Let us also include a normalization condition:

$$Tr(\ _i\ _j) = 2\ _{ij}$$

Then the matrices we need are the Pauli matrices:

$$_1 = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \qquad _2 = \begin{matrix} 0 & i \\ i & 0 \end{matrix} \qquad _3 = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \tag{2.1.3}$$

Then we will chose the group generators to be $F_i = \frac{1}{2}\ _i$. The group generated by these matrices is called the Special Unitary group of dimension 2, or SU(2) . In the fundamental representation, it is the group of 2 2 unitary matrices of unit determinant.

Recall from quantum mechanics that the structure constants are simply given by the Levi-Civita tensor:

$$f_{ijk} = \ _{ijk}$$

Only $_3$ is diagonal, and therefore the rank of SU(2) is 1. Then there is only one Casimir operator, and it is given by:

$$C^{(2)} = F_1^2 + F_2^2 + F_3^2$$

If we remove the Special condition (unit determinant), then the condition that $Tr\ _i = 0$ no longer need hold. Then there is one more matrix that satis es all other conditions and is still linearly independent from the others- the identity matrix:

$$_0 = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

Notice, however that this is invariant:

$$e^{i\ ^jF_j} e^{i\ ^0F_0} e^{\ i\ ^jF_j} = e^{i\ ^0F_0} \tag{2.1.4}$$

Therefore, the group generated by $e^{i\ ^0F_0}$ is an invariant ei80h1.9552 Tup

The whole group (without the special) is called U(2). We have found U(1) / U (2), SU(2) / U (2) and we can write:

$$U(2) = SU(2) \quad U(1) \tag{2.1.5}$$

This turns out to be true even in higher dimensional cases:

$$U(N) = SU(N) \quad U(1)$$

Sumarizing: $R = e^{i\frac{}{2}(\hbar\sim)}$ is the general form of an element of SU(2) in the fundamental representation. In this representation, $C^{(2)} = 2$, as you might expect if you followed the analogy from spin in quantum mechanics.

You know from quantum mechanics that there are other representations of spin (with di erent dimensions) that you can use, depending on what you want to describe - this change in representation is nothing more than calculating Clebsch-Gordon coe cients!

Example: what if you want to combine two spin-$\frac{1}{2}$ particles? Then you know that you get (3) spin-1 states and (1) spin-0 state. We can rewrite this in terms of representation theory: we'll call the fundamental representation 2, since it is two-dimensional. The spin-1 is three dimensional and the spin-0 is one dimensional, so we denote them by 3 and 1 respecitively. Then we can write the equation:

Intuitively we know that this metric is invariant under parity (space reversal), time reversal, spacial rotations, space and time translations and Lorentz transformations. Then we know that these operations should be represented in the Poincare group.

Ignore translations for now. You can prove to yourself that the matrices that represent the remaining operators must be orthogonal ($O^T = O^{-1}$); the group of n n orthogonal matrices is usually denoted by O(n). In this case, n = 4; however, there is a minus sign in the metric that changes things somewhat. Since three components have one sign, and one component has another, we usually denote this group as O(3,1) , and we call it the Lorentz group .

The transformations we are interested in are the proper orthochronous transformations- i.e.: the ones that keep causality and parity preserved. This is accomplished by im-

where $i, j = 1, 2, 3$, and $\mu, \nu = 0, 1, 2, 3$ and you can think of the wedge product as a generalization of the cross product (see Chapter 5). In this notation $J^{0i} \equiv K_i$ and $J^{ij} \equiv \epsilon_{ijk} S_k$. These six matrices are equivalent to the six above ones. Notice that, as before, the greek indices are not matrix indices.

2.2 Transformations

We wish to understand how transformations affect the equations of motion. The most important kinds of transformations are translations and rotations. These are called proper transformations. Other transformations might include parity, time reversal and charge inversion. We also might be interested in internal symmetries such as gauge transformations in E&M, for example.

When discussing the theory of transformations, it is useful to consider how objects transform under a very small change; such changes are called infinitesimal transformations. Let's consider these first. In what follows, I will be supressing indices (to make the notation simpler).

We want to invoke a transformation on a coordinate: $x \to x + \delta x$. Then we would expect a change in any function of x: $f(x) \to f(x) + \delta f$. What is the form of δf? If f represents a tensor, then it should have the same form as by definition.

Example: Consider an infinitesimal rotation about a point in the i direction. The transformation will look like:

$$x \to x + i\epsilon J_i x + O(\epsilon^2) = [1 + i\epsilon J_i + O(\epsilon^2)]x$$

where J_i is the generator of the rotation (angular momentum).

Recall that performing two rotations in the same direction is just multiplying the two rotations together. Then performing N infinitesimal rotations of size $\epsilon = a/N$, where a is a finite number, and ϵ is infinitesimal in the limit $N \to \infty$, we have:

$$\lim_{N\to\infty} \left(1 + \frac{iaJ_i}{N}\right)^N = e^{iaJ_i}) \quad x \quad \text{?!} \quad e^{iaJ_i}x \tag{2.2.9}$$

Hense we have constructed a finite transformation from an infinite number of infinitesimal ones. Notice that if you expand the exponential and keep only linear terms (for a very small) you reproduce the infinitesimal rotation.

2.3 Lagrangian Field Theory

Here is a brief overview of the important things you should know about Lagrangian field theory.

Recall in Lagrangian mechanics, you define a Lagrangian $L(q(t); \dot{q}(t))$ and an action $S = \int L dt$. Now we wish to decribe a Lagrangian density $L(\phi(x); \partial\phi(x))$ where $L = \int L d^3x$, and ϕ is a field which is a funtion of spacetime. The action can be written as $S = \int L d^4x$. The action completely specifies the theory.

The equations of motion can now be found, just as before by the condition $\delta S = 0$. This yields the new Euler-Lagrange equations:

$$\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)} = 0 \tag{2.3.10}$$

Notice that there is the sum over the index μ, following Einstein notation.

Example: $L = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2 \phi^2$ is the Lagrangian for a scalar field (Klein-Gordon field in relativistic quantum mechanics). Can you find the equations of motion? Here's the answer:

$$(\partial^2 + m^2)\phi = 0$$

Just as in classical mechanics, we can also go to the Hamiltonian formalism, by defining a momentum density π:

$$\pi = \frac{\partial L}{\partial \dot{\phi}}$$

where $\dot{\phi} = \frac{\partial \phi}{\partial t}$ and perform the Legendre transformation:

$$H(\phi(x), \pi(x)) = \pi(x)\dot{\phi}(x) - L \qquad H = \int d^3x \mathcal{H} \tag{2.3.11}$$

Notice that L is covariant, but H is not, since time becomes a special parameter via the momentum density. This is why we often prefer to use the Lagrangian formalism whenever working with relativistic mechanics. Nevertheless, this formalism is still generally useful since H is related to the energy in the field.

2.4 Noether's Theorem

A theory is said to be symmetric with respect to a transformation whenever the action doesn't change under that transformation. This implies that the Lagrangian can only change up to a total divergence (via Gauss's Theorem). To see this, notice that if the Lagrangian changes by a total divergence, we have:

$$S' = \int L d^4x + \int \partial J \, d^4x = S + \int d \, J$$

where the last surface integral is an integral over a surface at infinity. In order for quantities like energy to be finite, we often require that fields and their derivatives vanish on such a

surface. Thi implies that the above surface integral vanishes (or at least doesn't contribute to the Equations of Motion[3]). Thus we have

Noether's Theorem: For every symmetry in the theory, there corresponds a conserved current, and vice versa.

Proof: I will prove Noether's theorem for a scalar eld, but it is true in general for any type of eld. Let's consider the general transformation that acts on the eld according to:

$$! \quad +$$

where is a constant in nitesimal parameter[4]

energy conservation, etc. You can also consider more exotic internal symmetries; for example, gauge invariance in electrodynamics implies electric charge conservation.

Chapter 3

Geometry I: Flat Space

The most important aspect of geometry is the notion of distance. What does it mean for two points in a space to be \close" to each other? This notion is what makes geometry distinct from topology and other areas of abstract mathematics. In this chapter, I want to de ne some important quantities that appear again and again in geometric descriptions of physics, and give some examples.

In this chapter we will always be working in at space (no curvature). In the next chapter, we will introduce curvature and see how it a ects our results.

3.1 THE Tensor: g

Like transformations, we can de ne distance in nitesimally and work our way up from there. Consider two points in space that are separated by in nitesimal distances dx, dy and dz

3.1.1 Curvilinear Coordinates

In spherical coordinates, the directions are given by $(dr; d\ ; d\ )$, and the metric is given by:

$$ds^2 = dr^2 + r^2 d\ ^2 + r^2 \sin^2\ d\ ^2$$

so the metric tensor is therefore:

$$[g\ ] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \end{pmatrix}$$

$$ds^2 = \frac{1}{4}\left(1+\frac{}{}\right)d^2 + \left(1+\frac{}{}\right)d^2 + d^2$$

therefore:

$$[g\ ] = \begin{pmatrix} \frac{1}{4}\left(1+\frac{}{}\right) & 0 & 0 \\ 0 & \frac{1}{4}\left(1+\frac{}{}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Parabolic coordinates describe families of concentric paraboloids symmetric about the z-axis. They are very useful in problems with the appropriate symmetry, such as parabolic mirrors.

As you might have guessed, there are also hyperbolic coordinates and elliptical coordinates, but I won't get into those here. There are even more complicated coordinate systems that are useful (toroidal coordinates, etc).

3.2 Di erential Volumes and the Laplacian

3.2.1 Jacobians

If you transform your coordinates, you will modify your di erential volume element by a factor called the Jacobian :

$$y\ = y\ (x\ )\ )\quad d^n x = J\ d^n y$$

where in general, $J\ = \det\ \frac{@\bar{x}^a}{@y}$

To nd the general Jacobian, let us consider a transformation from Cartesian coordinates (x^a) to general coordinates ($y\ $) and ask how the metric transforms. Remembering that the metric is a $(0;2)$ tensor and $g_{ab} = {}_{ab}$ we have:

$$g\ = {}_{ab}\frac{@\bar{x}^a}{@y}\frac{@\bar{x}^b}{@y} \tag{3.2.7}$$

Now take the determinant of both sides, and de ning $g\ \ \det(g\ )$ we nd that $J = \sqrt{\bar{g}}$, so in general:

$$d^n x = \sqrt{\bar{g}}\, d^n y$$

.

3.2.2 Veilbeins - a Prelude to Curvature

I will talk about curvature in the next chapter, but now would be a good time to introduce a very useful tool called the vielbein (e_a):

Consider a point p with coordinates x on a manifold (generally a manifold with curvature, which we'll talk about next Chapter). At every point p 2 M, there is a \tangent space" { this is basically the at plane tangent to the manifold at the point p. Give coordinates on the tangent plane ^{a} such that the point p is mapped to the origin. The key point to notice is that the latin index is for at space while the greek index is for curved space.

Now consider a point p on the manifold, and the tangent space at p. Then there is a (linear) mapping between the points near the origin of the tangent plane (with coordinates ^{a}) to a neighborhood of p on the manifold (with coordinates x). This linear mapping is given by a matrix called the vielbein :

$$e_a \quad \frac{@x}{@^a}$$

This allows us to write Equation (3.2.7) as:

$$g \quad = \quad _{ab}e^a e^b$$

So the vielbein is, in some sense, the square rootof the metric! In particular, the above analysis shows that the Jacobian is just e, the determinant of the vielbein. There are many cases in physics where this formalism is very useful[2]. I won't mention any of it here, but at least you have seen it.

3.2.3 Laplacians

We can nd the Laplacian of a scalar eld in a very clever way using the Variational principle in electrostatics. For our functional, consider that we wish to minimize the energy of the system, and in electrostatics, that is (up to constants) the integral of E^2. So using our scalar potential, let's de ne the energy functional in cartesian coordinates:

$$W[\ ] = \int d^n x (r\ )^2$$

where $d^n x$ is the n dimensional di erential volume element.

We can nd the extremum of this functional W [ ] = 0 and see what happens. At rst, notice that an integration by parts lets you rewrite the integrand as r ^{2} , and the variation of that is simply r ^{2} . Now change coordinates using the above prescription and repeat the variation. We set the two expressions equal to each other,

[2]For example, when spin is involved.

way to think of this (and perhaps the better and more general way) is to replace $g \to j$ gj in all of our equations. Then there will never be a problem.

Just for completeness, the metric tensor of Minkowski spacetime is:

$$= \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is called the \mostly-minus" prescription. Sometimes people will use the \mostly plus" prescription which is the negative of this. You must be very careful which convention is being used - particle physicists often use the minus prescription I use here, while GR people tend to use the mostly plus prescription. But sometimes they switch! Of course, physics doesn't change, but the formulas might pick up random minus signs, so be careful.

Chapter 4

Geometry II: Curved Space

Consider the planet Earth as a sphere. We know that in Euclidean geometry, if two lines are perpendicular to another line, then they are necessarely parallel to each other (they never intersect). Yet a counterexample of this is longitude lines: two lines of longitude are always perpendicular to the equator, and so they are \parallel"; yet they meet at two points: the north and south poles. Obviously, something is wrong with Euclidean geometry on a sphere.

In everything that we have been doing, we have been assuming that the space we live in was at. Even when we were using curvilinear coordinates, we were still talking about at space. The situation is fundamentally di erent when space itself is curved. It is here where di erential geometry truly becomes interesting!

There is a very deep theorem in mathematics that states that any manifold of any dimension and any curvature can always be embedded in a at space of some larger dimension. For example, a sphere (which is two dimensional) can be embedded into Euclidean 3-space. So in principle, we can avoid the issue of curvature completely if we think in enough dimensions. But this is not practical or necessary in general, and curvature plays a very important role in mathematical physics.

Di erential geometry and curved space have their most obvious role in general relativity (GR), where gravity manifests as a warping of spacetime. However, it also has applications in several other branches of mathematical physics, including (but not limited to) optics, solid-state physics and quantum eld theory. So it might be useful to see some of the basics in one place. Hence this chapter.

Di erential geometry is truly a deep and profound subject, and I recommend anyone interested to study it either in a class or on your own (I took three classes in it as an undergraduate and I'm still trying to learn it!) But I do not plan to teach di erential geometry here; only give an overview and a source for common equations you might come across in physics. For a deeper understanding, you have to go to a textbook. I give some recomendations in the bibliography.

4.1 Connections and the Covariant Derivative

One of the biggest problems with geometry on curved spaces is the notion of the derivative of a vector eld. When you take the derivative, you are actually taking the di erence of two

vectors that are separated by an in tiesimal distance:

where the matrices A are the coordinate transformation matrices de ned in Chapter 1.

It seems reasonable that this object might be like the generalization of the derivative. However, it turns out that we're not quite done - these three axioms de ne a connection, but they do not de ne it uniquely. If we want to de ne something like a derivative, we need to invoke more axioms. We usually chose the following two:

4. $r_X g \quad 0 \qquad 8X$

5. $\Gamma^i_{jk} = \Gamma^i_{kj}$

The rst of these is often put into words as: \The connection is compatible with the metric." It basically means that the derivative of a product makes sense. The second property is that the connection is symmetric or torsionless . These two additional properties, along with the three already included, are necessary and su cient to de ne a unique connection called the Riemann Connection , or the Levi-Civita Connection . Before we proceed, notice that this is by no means the only choice I could have made. There are many theori

Remembering our easy notation of Chapter 1, it is sometimes easier to write the di erential coordinate index with a semi-colon:

$$\boxed{A^k_{;i} = A^k_{;i} + A^j \ {}^k_{ij}} \tag{4.1.5}$$

What if I wanted to take the covariant derivative of a tensor that is higher than rank-1? Let's consider the covariant derivative of the diadic:

$$(A_i B_j)_{;k} = A_{i;k} B_j + A_i B_{j;k} = (A_i B_j)_{;k} + (\ {}^m_{ik} A_m) B_j + (\ {}^m_{jk} B_m) A_i$$

where I have skipped some straightforward algebra. This, along with Equation (4.1.4), should give us enough intuition to express the covariant derivative on an arbitrary rank tensor:

$$\boxed{(T^{i_1 i_2 \ \ i_n})_{;j} = (T^{i_1 i_2 \ \ i_n})_{;j} + \sum_{=1}^{n} \ {}^{i}_{j} \ T^{i_1 \ \ i \ \ _1 i \ _{+1} \ \ i_n}} \tag{4.1.6}$$

In other words { for a rank-n tensor, we get a term for each contravariant index. Similarly there would be a minus term for each covariant index. By the way, a quick corollary to this would be:

$$_{;i} = \ _{;i}$$

for a scalar quantity . This should be so, since the Christo el symbols only appeared when we had basis vectors to worry about.

Now that we have the covariant derivative under our belts, we can use it to nd a useful expression for the connection coe cients. Take the covariant derivative of the metric tensor three times (each with a di erent index), and use the property of compatability to get:

$$\boxed{\ = \frac{1}{2} g \ \ [g_{;} \ \ \ g_{;} \ \ + g_{;} \ ]} \tag{4.1.7}$$

As a nal exercise, consider a parametrized curve on a manifold M, w(s) : I ! M, where I is the unit interval [0,1]. Then we can take the parametrized covariant derivative by resorting to the chain rule:

$$\frac{Dw^j}{ds} = D_i w^j \frac{dx^i}{ds} = \frac{@\dot{w}}{@\dot{x}} \frac{dx^i}{ds} + (\ {}^j_{ik} w^k) \frac{dx^i}{ds}$$

Combining terms:

$$\boxed{Dw^j \ \ ‘}$$

4.2 Parallel Transport and Geodesics

Newton's laws of motion! General relativity suggests that in the absense of inertial forces, a particle moves along a geodesic, so its equation of motion is given by Equation (4.2.9). Gravity manifests itself as the curvature of space, i.e.: the connection coe cients. So we can think of the terms in Equation (4.2.9) as source terms for the gravitational eld in Newton's equations of motion. If we wished to include other inertial forces (such as an electric eld, for instance), we would modify the right hand side of Equation (4.2.9).

Additionally, a geodesic is the is the path of shortest distance: this can be proved using variational calculus techniques. So this equation tells us that particles like to travel along the path the minimizes the distance travelled. In at space (no forces), that's a straight line; but in curved space (when gravity is present, for example) then in can be more complicated.

4.3 Curvature- The Riemann Tensor

Now that we have worked out some details on parallel transport and geodesics, let's see what else the covariant derivative can give us. One question that analysts like to ask is \Are mixed partials equal?" It turns out that understanding the answer to this question gives us a rigorous way to quantify the curvature of the manifold.

Consider the commutator of two covariant derivatives acting on a vector[5]:

$$A_{;\ ;} \quad A_{;\ ;} = A \ R^{:}$$

where

$$\boxed{R^{:}_{\quad ;\quad ;\quad +}} \tag{4.3.10}$$

$R^{:}$ is a tensor, called the Riemann-Christo el Curvature Tensor . We know it's a tensor from the quotient rule of tensors: if a tensor equals another tensor contracted with another object, then that other object must be a tensor. There are a number of symmetries[6]:

1. $R \quad = \quad R$
2. $R \quad = \quad R$
3. $R \quad = +R$
4. $R^{:}_{\quad} + R^{:}_{\quad} + R^{:}_{\quad} = 0$
5. $R^{:}_{\quad ;} + R^{:}_{\quad ;} + R^{:}_{\quad ;} = 0$

The rst four symmetries are manifest from Equation (4.3.10); the last item is known as the Bianchi Identity . These symmetries have the e ect of putting some powerful constraits on the number of independent quantities in the curvature tensor. Without any symmetries, there would be d^4 numbers, where d is the dimension of our manifold. However, with all of these symmetries, there are:

[5] I'm not going to prove this: you do it!

[6] $R^{:}_{\quad} = g \quad R$

$$\frac{d^2(d^2 \quad 1)}{12} \tag{4.3.11}$$

independent quantities. Notice that the rank of the Riemann-Christo el tensor is always 4, despite the dimension of the manifold.

The Bianchi Identity is a di erent sort of relation in the sense that it involves derivatives. There is a similar equation in Maxwell's electrodynamics:

$$F_{\ ;\ } + F_{\ ;\ } + F_{\ ;\ } = 0$$

where F is the eld strength tensor (see Jackson or Gri ths). These equations are very closely related to the topic of \Gauge Invariance" { we will talk about that in the next chapter.

We can try to construct lower rank tensors from this monster in Equation (4.3.10), but thanks to the symmetries, there are only a few nonzero contractions, and up to sign, they all yield the same thing. We de ne the rank-2 tensor:

$$R \qquad R_{:} \tag{4.3.12}$$

It is called the Ricci tensor . It is a symmetric tensor. Any other nonzero contraction would give the same object up to a sign. You should prove this for yourself. Finally we can take the trace of Equation (4.3.12) to get the Ricci scalar :

$$R \quad R \tag{4.3.13}$$

R is sometimes called the scalar curvature . It is a measure of the intrisic curvature of the manifold.

Finally, I would like to mention an interesting curiosity. Take a good look at Equation (4.3.10). Einstein used this tensor to describe gravity. Notice that it has derivatives of the connection coe cients, and is a tensor. It is, in a sense, a strange kind of curl of the connection coe cients. Now look at Equation (4.1.2) - this says that the connection coe cients transform under coordinate transformations with an inhomogeneous term. Does this look like anything we know? What about the vector potential of electrodynamics? That has an inhomogeneous term when you perform a gauge transformation (A ! A + r), and yet when you take its curl to get B, the resulting eld is gauge invariant. Here, the \gauge transformation" is actually a coordinate transformation (technically, a \di eomorphism"), and the Riemann tensor transforms \covariantly" (which is a fancy word for \properly") under these transformations, even though the connection coe cients do not. You can see the parallel: you might think of the connection coe cients as \gravity vector potentials", and the curvature tensor as the \gravitational eld". Then we have something like a gauge theory of gravity! Unfortunately, things are more complicated than that. However, this is getting us too far a eld, so I'll stop now.

4.3.1 Special Case: d = 2

Before leaving di erential geometry behind, I wanted to talk a little about a very special case of mathematics - two dimensional manifolds. In one dimension, the Riemann tensor

is not well-de ned, and we do not need any of this formalism to discuss geometry; in fact, one dimensional di erential geometry is nothing more than vector calculus! In three or more dimensions, the Riemann tensor is complicated, and it is much more di cult to say anything quantitative and general about your manifold. But in the special case of two dimensions, the Riemann tensor has only one component (see Equation (4.3.11)), and that is the Ricci scalar curvature. Here, we can talk about curvature in a very quantitative matter.

To understand how this works, we need to state a few de nitions. A two dimensional Riemannian manifold is often simply referred to as a \surface".

De nition: Consider a curve parametrized by (s) living on a surface M. The the curvature of the curve is given by:

$$k(s) \quad j \quad ^{00}(s)j$$

If n is the normal vector to , and N is the normal to the surface, then we de ne the angle:

$$\cos \quad _{(s)} \quad h\, n(s); N(s)i$$

Then

$$k_n(s) \quad k(s)\cos \quad _{(s)}$$

is called the normal curvature along the path (s).

need no longer be topologically equivalent in higher dimensions, and we know of no single quantity that does classify manifolds in this way. This is way beyond the scope of this review, so I will stop now. Let me just say that this is one of the most active areas in modern mathematics research. If you are at all interested, I strongly encourage you to check it out!

Chapter 5

Di erential Forms

Di erential forms are, without a doubt, one of the most beautiful inventions in all of mathematics! Their elegance and simplicity are without bound. Nevertheless, like anything good, it takes a long time before they become natural and second nature to the student. Whether

De nition:

example, acording to this de nition, the Christo el symbols of di erential geometry are $(1;2)$ tensors. So you must be sure you know which de nition people are using.

That's enough abstraction for one day. Let us consider a concrete example. Consider the vector space R^n and the set of multilinear transformations that take k vectors to a number:

$$: \underbrace{R^n \quad R^n}_{k \text{ times}} \{z \quad R^n\} ! \quad R \tag{5.1.2}$$

with the additional property that they are \alternating", i.e.:

$$(: : : ; x; y; : : :) = \quad (: : : ; y; x; : : :) \tag{5.1.3}$$

When takes k vectors in R^n, then is called a k-form . The set of all k-forms is denoted $^k(R^n)$. Note that this set is also a vector space. The dual space is an example of this, hence the notation.

In linear algebra, we know that every vector space has a basis, and the dimension of the

5.2 Form Operations

5.2.1 Wedge Product

In order to de ne the standard basis for $^k(R^n)$, we needed to introduce a new operation, called the wedge product ($\wedge$). Let's look more carefully at this new form of multiplication. As a \pseudo-de nition", consider two 1-forms $!; \quad 2 \quad ^1(R^n)$; then let us de ne an element in $^2(R^n)$ by the following:

$$! \wedge \quad (\vec{v}; \vec{w}) = ! (\vec{v}) \quad (\vec{w}) \quad ! (\vec{w}) \quad (\vec{v}) \tag{5.2.7}$$

Notice that there are two immediate corollaries:

$$! \wedge \quad = \quad \wedge ! \tag{5.2.8}$$

$$! \wedge ! \quad = \quad 0 \qquad 8! \quad 2 \quad ^k(R^n) \tag{5.2.9}$$

So the wedge product de nes a noncommutative product of forms. Notice that this is di erent from the ordinary tensor product:

$$! \qquad (\vec{v}; \vec{w}) \quad ! (\vec{v}) \quad (\vec{w}) \tag{5.2.10}$$

Looking at this as an antisymmetric product on forms, you might be reminded of the cross product. This is exactly right: the cross-product of two vectors is the same thing as a wedge product of two 1-forms. We will prove this explicitly soon.

Now that we have talked about the speci c case of 1-forms, we can generalize to the wedge product of any forms:

De nition: Let $! \quad 2 \quad ^k(R^n); \quad 2 \quad ^l(R^n)$. Then we de ne $! \wedge \quad 2 \quad ^{k+l}(R^n)$ as the sum of all the antisymmetric combinations of wedge products. This product is

1. Distributive: $(dx^1 + dx^2) \wedge dx^3 = dx^1 \wedge dx^2 + dx^1 \wedge dx^3$
2. Associative: $(dx^1 \wedge dx^2) \wedge dx^3 = dx^1 \wedge (dx^2 \wedge dx^3) = dx^1 \wedge dx^2 \wedge dx^3$
3. Skew-commutative: $! \wedge \quad = (\quad 1)^{kl} \quad \wedge !$

Notice that for $k + l > n$ the space is trivial, so $\wedge ! = 0$.

Before going any further, let's do some examples. Consider $!; \quad 2 \quad ^1(R^3)$, i.e.: 1-forms in 3-space. Writing it out explicitly, we have:

$$\begin{aligned} ! &= a_1 dx^1 + a_2 dx^2 + a_3 dx^3 \\ &= b_1 dx^1 + b_2 dx^2 + b_3 dx^3 \\ ! \wedge &= (a_1 dx^1 + a_2 dx^2 + a_3 dx^3) \wedge (b_1 dx^1 + b_2 dx^2 + b_3 dx^3) \\ &= (a_1 b_2 \quad a_2 b_1) dx^1 \wedge dx^2 + (a_1 b_3 \quad a_3 b_1) dx^1 \wedge dx^3 + (a_2 b_3 \quad a_3 b_2) dx^2 \wedge dx^3 \end{aligned}$$

Notice how I've been careful to keep track of minus signs. Also notice that this looks very similar to a cross product, as stated earlier. What about taking the wedge product of two 1-forms in R^4:

$$\begin{aligned} &= a_1dx^1 + a_2dx^2 + a_3dx^3 + a_4dx^4 \\ &= b_1dx^1 + b_2dx^2 + b_3dx^3 + b_4dx^4 \\ \wedge \quad &= (a_1b_2 - a_2b_1)dx^1 \wedge dx^2 + (a_1b_3 - a_3b_1)dx^1 \wedge dx^3 + (a_1b_4 - a_4b_1)dx^1 \wedge dx^4 \\ &\quad + (a_2b_3 - a_3b_2)dx^2 \wedge dx^3 + (a_2b_4 - a_4b_2)dx^2 \wedge dx^4 + (a_3b_4 - a_4b_3)dx^3 \wedge dx^4 \end{aligned}$$

Note that this does not look like a \cross product"; it is a six-dimensional object in R^4. However, if you took another wedge product with another 1-form, you would get a 3-form in R^4 (which is four-dimensional), and that would give you something like a four-dimensional cross product. In general, taking the wedge product of $(n - 1)$ 1-forms in R^n will give you something analogous to a cross-product. We'll get back to this later.

As a nal example, what if you took the wedge product of a 1-form and a 2-form in R^3:

$$= c_1dx^1 + c_2dx^2 + c_3dx^3 \tag{5.2.11}$$

$$= L_{12}dx^1 \wedge dx^2 + L_{13}dx^1 \wedge dx^3 + L_{23}dx^2 \wedge dx^3 \tag{5.2.12}$$

$$\wedge \quad = (c_1L_{23} - c_2L_{13} + c_3L_{12})dx^1 \wedge dx^2 \wedge dx^3 \tag{5.2.13}$$

Notice that if we identify $= ! \wedge$ from above, then we have shown that the triple wedge product of 1-forms in R^3 is just:

$$! \wedge \wedge = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} dx^1 \wedge dx^2 \wedge dx^3 \tag{5.2.14}$$

In fact, the wedge product of n 1-forms in R^n is always such a determinant. This is a very important fact that will come up again and again.

Next, we can ask how a general wedge product of forms acts on vectors? The answer is that the form farthest to the left acts on the vector, and then you must permute the form so that all the 1-forms act on the vector. As an example, consider a 2-form acting on a vector:

$$dx \wedge dy(\vec{v}) = dx(\vec{v})dy - dy(\vec{v})dx = v_x dy - v_y dx$$

The minus sign comes from ipping the dx and the dy.

Before moving on, let us conclude this section with a de nition and a theorem:

De nition: A simple k-form is a \monomial" k-form; that is, there is no addition.

Theorem: All k-forms can be written as a linear combination of simple k-forms.

Proof: The proof of this theorem is straightforward: a simple k-form is just a basis element, possibly multiplied by a number. Then since we are dealing with a vector space, any element of $^k(R^n)$ can be written as a linear combination of these basis elements. QED.

In R^3:

$$\begin{array}{llll} dx^1 = dx^2 \wedge dx^3 & & (dx^1 \wedge dx^2) = dx^3 \\ dx^2 = dx^3 \wedge dx^1 & & (dx^2 \wedge dx^3) = dx^1 \\ dx^3 = dx^1 \wedge dx^2 & & (dx^3 \wedge dx^1) = dx^2 \end{array}$$

In R^4:

$$\begin{array}{llll} dx^1 = +dx^2 \wedge dx^3 \wedge dx^4 & (dx^1 \wedge dx^2) = +dx^3 \wedge dx^4 & (dx^1 \wedge dx^2 \wedge dx^3) = +dx^4 \\ dx^2 = dx^3 \wedge dx^4 \wedge dx^1 & (dx^1 \wedge dx^3) = dx^2 \wedge dx^4 & (dx^1 \wedge dx^2 \wedge dx^4) = dx^3 \\ dx^3 = +dx^4 \wedge dx^1 \wedge dx^2 & (dx^1 \wedge dx^4) = +dx^2 \wedge dx^3 & (dx^1 \wedge dx^3 \wedge dx^4) = +dx^2 \\ dx^4 = dx^1 \wedge dx^2 \wedge dx^3 & (dx^2 \wedge dx^3) = +dx^1 \wedge dx^4 & (dx^2 \wedge dx^3 \wedge dx^4) = dx^1 \\ & (dx^2 \wedge dx^4) = dx^1 \wedge dx^3 & \\ & (dx^3 \wedge dx^4) = +dx^1 \wedge dx^2 & \end{array}$$

We can see a pattern here if we remember how the Levi-Civita simbol works. For any form in $^k(R^n)$:

$$= \quad _{1\cdots k} dx^1 \wedge \quad \wedge dx^k)$$

$$\boxed{= \frac{1}{(n \quad k)!} \quad _{1\cdots n} \quad ^{1\cdots k} dx^{k+1} \wedge \quad \wedge dx^n} \qquad (5.2.16)$$

or

$$\boxed{_{1\cdots n \quad k} = \frac{1}{(n \quad k)!} \quad _{1\cdots k \; 1\cdots n \quad k} \quad ^{1\cdots k}} \qquad (5.2.17)$$

From the above, we can now present a theorem:

Theorem: For any k-form $! \; 2 \quad ^k(R^n)$,

basis of the three vectors you chose! In other words, di erential forms automatically give us the vector calculus result:

$$d^n x' = J\, d^n x \tag{5.2.19}$$

5.2.4 Evaluating k-Forms

We have done a lot of de ning, but not too much calculating. You might be asking at this

5.2.5 Generalized Cross Product

Eariler, I mentioned that we can de ne a generalized cross product in R^n by wedging together $(n \quad 1)$ 1-forms. Let's see this explicitly:

In R^n, take $n \quad 1$ vectors, and tilde them so they are now 1-forms: $f\tilde{v}_1; : : : ; \tilde{v}_{n \quad 1}g$. Now wedge these forms together to give you an $(n \quad 1)$-form, and take the Hodge star of the product to give us a 1-form. Finally tilde the whole thing to give us an n-vector, and we have de ned a general product of vectors:

$$\boxed{\tilde{v}_1 \quad \tilde{v}_{n \quad 1} \quad [\ (v_1 \wedge \wedge \quad \wedge v_{n \quad 1})] \ 2 \ R^n} \tag{5.2.20}$$

5.3 Exterior Calculus

A k-form eld is a k-form whose coe cients depend on the coordinates. This is exactly analogous to a vector eld in vector calculus. So, if we have a vector eld in R^3:

$$\begin{aligned} \tilde{v}(x) &= v^1(x)\tilde{e}_1 + v^2(x)\tilde{e}_2 + v^3(x)\tilde{e}_3 \\ v(x) &= v_1(x)dx^1 + v_2(x)dx^2 + v_3(x)dx^3 \end{aligned}$$

5.3.1 Exterior Derivative

Now that we have a notion of \function", let's see what we can do in the way of calculus. We can de ne an operator

$$d : \ ^k(R^n) \ ! \quad ^{k+1}(R^n)$$

with the all-important property:

$$d^2 = dd \quad 0 \tag{5.3.21}$$

By construction we will let d act on forms in the following way:

$$d(a(x)dx) = da(x) \wedge dx \tag{5.3.22}$$

Let's look at some examples.

$$f(x) \ 2 \ ^0(R^3) \ ! \quad df(x) = \frac{@f}{@x^1} \ dx^1 + \frac{@f}{@x^2} \ dx^2 + \frac{@f}{@x^3} \ dx^3 \ 2 \ ^1(R^3)$$

This is just the gradient of a function! How about the curl?

$$
\begin{aligned}
\omega &= a_1 dx^1 + a_2 dx^2 + a_3 dx^3 \\
d\omega &= \frac{\partial a_1}{\partial x^1} dx^1 + \frac{\partial a_1}{\partial x^2} dx^2 + \frac{\partial a_1}{\partial x^3} dx^3 \wedge dx^1 \\
&\quad + \frac{\partial a_2}{\partial x^1} dx^1 + \frac{\partial a_2}{\partial x^2} dx^2 + \frac{\partial a_2}{\partial x^3} dx^3 \wedge dx^2 \\
&\quad + \frac{\partial a_3}{\partial x^1} dx^1 + \frac{\partial a_3}{\partial x^2} dx^2 + \frac{\partial a_3}{\partial x^3} dx^3 \wedge dx^3 \\
&= \frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2} dx^1 \wedge dx^2 + \frac{\partial a_3}{\partial x^1} - \frac{\partial a_1}{\partial x^3} dx^1 \wedge dx^3 + \frac{\partial a_3}{\partial x^2} - \frac{\partial a_2}{\partial x^3} dx^2 \wedge dx^3
\end{aligned}
$$

Notice that terms like $\frac{\partial a_1}{\partial x^1} dx^1 \wedge dx^1$ vanish immediately because of the wedge product. Again we see how the beauty of differential forms notation pays off!

5.3.2 Formulas from Vector Calculus

At this point we can immediately write down some expressions. Some of them I have shown. Can you prove the rest of them? In all cases, f is any (differentiable) function, and $\vec{v}$ is a vector field.

$$
\begin{aligned}
df &= \nabla f && (5.3.23)\\
g_{\vec{v}} &= \nabla \cdot \vec{v} \quad (\text{in } R^3) && (5.3.24)\\
d\,\vec{v} &= \nabla \times \vec{v} && (5.3.25)\\
d\,df &= \nabla^2 f && (5.3.26)
\end{aligned}
$$

Notice that even though $dd = 0$, $d\,d\vec{v} \neq 0$ in general!

5.3.3 Orthonormal Coordinates

Differential forms allow us to do calculations without ever refering to a coordinate system. However, sooner or later we will want to get our answers back into a coordinate frame. This could be tricky.

The key point to remember is that each of these dx^i is an orthonormal coordinate[2]. Therefore y

[2] x;327 T

Hoa-27(er27(v)27(e)-,32645727(8y)23.4k)2n27(w)-3554thatdx]TJ/F29 7.9701 Tf 9.963 2.338 Td [(2)]TJ/F17 11.9552 T

$$
\begin{aligned}
df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\
df &= \frac{\partial f}{\partial x}dy - \frac{\partial f}{\partial y}dx \\
d\,df &= \frac{\partial^2 f}{\partial x^2}dx \wedge dy - \frac{\partial^2 f}{\partial y^2}dy \wedge dx \\
&= \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy \\
d\,df &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}
\end{aligned}
$$

All is well with the world. But what if I wanted to do my work in polar coordinates? Then my coordinate system is $(x^1 = r; x^2 = \ )$, but the orthonormal polar coordinate basis vectors are $\mathbf{e}_1 = \hat{r}; \mathbf{e}_2 = \frac{\hat{\ }}{r}$, and the dual basis is $(dr; r\,d\ )$ (see previous footnote). So:

$$
\begin{aligned}
df &= \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial\ }d\ \frac{r}{r} = \frac{\partial f}{\partial r}dr + \frac{1}{r}\frac{\partial f}{\partial\ }(r\,d\ ) \\
df &= \frac{\partial f}{\partial r}(r\,d\ ) - \frac{1}{r}\frac{\partial f}{\partial\ }dr \\
d\,df &= \frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) dr \wedge d\ - \frac{\partial}{\partial\ }\left(\frac{1}{r}\frac{\partial f}{\partial\ }\right) d\ \wedge dr \quad \frac{r}{r} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\ ^2}\ dr \wedge (r\,d\ ) \\
d^{\ 2} &+ \partial)\,r\,d
\end{aligned}
$$

5.4.1 Evaluating k-form Fields

First, a de nition:

De nition: An Oriented k-Parallelogram , denoted by $P_x^0\{\vec{v}_i\}$, $(i = 1;\ldots;k)$, is a k-parallelogram spanned by the k n-vectors $\{\vec{v}_i\}$ with basepoint x. $P_x^0\{\vec{v}_i\}$ is antisymmetric under $\vec{v}_i$ exchange.

Oriented k-Parallelograms allow you to think geometrically about evaluating k-forms, but as far as paperwork goes, they are just a way to keep track of what vectors are what. We use them explicitly to evaluate k-form elds at a point. Let's see how that works:

Let's evaluate the 2-form eld $= \cos(xz)dx \wedge dy \in \Omega^2(R^3)$. We'll evaluate it on the oriented 2-parallelogram spanned by the 3-vectors $\vec{v}_1 = (1;0;1)$ and $\vec{v}_2 = (2;2;3)$. We'll evaluate it at two points: $(1;2;\ )$ and $(\frac{1}{2};2;\ )$:

1. $$P^0_{\begin{pmatrix}1\\2\\\end{pmatrix}}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix};\begin{pmatrix}2\\2\\3\end{pmatrix}\right\} = \cos(1$$

Like in vector calculus, we can de ne the integral formally in terms of Riemann sums - yuck! Let me just say that this is an equivalent de nition; the more curious students can go prove it.

Let's do two examples to get the hang of it:

1. Consider $2 \ ^1(R^2)$ and integrate over the map:

$$(u) = \begin{matrix} R\cos u \\ R\sin u \end{matrix}$$

over the region $A = [0; \ ]$, (> 0). Let $= xdy \quad ydx$:

$$\begin{aligned} \int_{(A)} &= \int_{[0; \ ]} (xdy \quad ydx) \\ &= \int_0^{} \begin{matrix} R\cos u \\ R\sin u \end{matrix}^{1} \begin{matrix} R\sin u \\ R\cos u \end{matrix} du \\ &= \int_0^{} du[(R\cos u)(R\cos u) \quad (R\sin u)(\ R\sin u)] \\ &= \int_0^{} duR^2[\cos^2 u + \sin^2 u] \\ &= R^2 \end{aligned}$$

2. Consider $= dx \wedge dy + ydx \wedge dz \ 2 \ ^2(R^3)$, and the map:

$$\begin{matrix} s \\ t \end{matrix} = \begin{matrix} s + t \\ s^2 \\ t^2 \end{matrix}$$

over the region $C = f \begin{matrix} s \\ t \end{matrix} \ j0 \quad s \quad 1; 0 \quad t \quad 1g$:

$$\int_{(C)} = \int_0^1 \int_0^1 (dx \wedge dy + ydx \wedge dz) \begin{matrix} s + t \\ s^2 \\ t^2 \end{matrix}^{1} \left\{ \begin{matrix} 0 \\ 2s \\ 0 \end{matrix}^{1} ; \begin{matrix} 1 \\ 0 \\ 2t \end{matrix}^{1} \right\} = \int_0^1 \int_0^1 C$$

5.4.3 Stokes' Theorem

Now we can move on to present one of if not the most important theorems that di erential forms has to o er- Stokes' Theorem. I will present it correctly; do not be overly concerned with all the hypotheses; it su ces that the area you are integrating over has to be appropriately \nice".

Theorem: Let X be a compact, piece-with-boundary of a (k+1)-dimensional oriented manifold M $\subset$ R^n. Give the boundary of X (denoted by @X) the proper orientation, and consider a k-form eld $\in$ $^k(R^n)$ de ned on a neighborhood of X. Then:

$$\int_{@X} = \int_X d \tag{5.4.28}$$

What is this theorem saying? My old calculus professor used to call it the \Jumping-d theorem", since the \d" jumps from the manifold to the form. In words, this theorem says that the integral of a form over the boundary of a su ciently nice manifold is the same thing as the integral of the derivative of the form over the whole mainfold itself.

5.5 Forms and Electrodynamics

As a nale for di erential forms, I thought it would be nice to summarize brie y how one uses forms in theories such as E&M. Recall that in covariant electrodynamics, we have an antisymmetric, rank-2 4-tensor known as the \Field Strength" tensor:

$$F \quad F \; dx \quad dx \tag{5.5.29}$$

As we have seen, it is always possible to use 2-forms instead of (antisymmetric) tensors, and we can rewrite the above tensor as a di erential form in $^2(M_4$

$$\int_{\partial} F = \int_{} dF = 0 \tag{5.5.34}$$

$$\int_{@} F = \int_{} d \ F = 4 \ (\text{charge}) \tag{5.5.35}$$

The first of these equations says that the total flux of F through a closed region of space-time is zero; the second equation says that the total flux of F through a closed region of space-time is proportional to the amount of charge in that region. Notice that this description never mentions coordinate systems: once again, differential forms has allowed us to describe physics and geometry without ever referring to a coordinate system! Notice the similarity of this equation with the Gauss-Bonnet Theorem – in gauge theories, you may think of the field strength as a curvature to some space!

The final step in studying electrodynamics is to notice that Equation (5.5.32) suggests something. Recall that if a differential form was the exterior derivative of some other form, then it's exterior derivative was necessarely zero, via Equation (5.3.21). Is the converse true? Namely, if you have a form whose exterior derivative is zero, can it be written as the exterior derivative of some other form?

Your first impulse might be to say "yes": surely if the form is an exterior derivative, your condition is satisfied. But it turns out that the question is a little more subtle than that. For I can give you a form which is not an exterior derivative of another form, and yet still has vanishing exterior derivative! This is a famous case of "necessity" versus "sufficiency" in mathematics. That a form is an exterior derivative is sufficient for its exterior derivative to vanish, but not necessary.

The key point is that this property depends not on the differential forms themselves, but on the global properties of the space they live in! It turns out that M_4 does indeed have the property of necessity; specifically, it is simply connected. Therefore it is safe to assume a la Equation (5.5.32) that we can write:

$$F = dA \tag{5.5.36}$$

for some 1-form A. So indeed, electrodynamics in flat space can be described by a 4-vector potential. But do be careful not to jump to any conclusions before you know the properties of the universe you are describing!

Finally, we can consider what happens if we let $A \to A + d$, where is any (reasonably well-behaved) function. Then by just plugging into Equation (5.5.36), with the help of Equation (5.3.21), we see that F remains the same. Hense, the 4-vector potential is only defined up to a 4-gradient; this is exactly what gauge invariance tells us should happen!

This gives a very powerful geometric intuition for our field-strength, and therefore for electricity and magnetism. You can do similar analyses for any gauge field, and by adjusting the manifold, you can alter your theory to include gravity, extra dimensions, strings, or whatever you want! This is one of the biggest reasons why differential forms are so useful to physicists.

Chapter 6

Complex Analysis

In this chapter, I would like to present some key results that are used in analytic work. There is of course much more material that what I present here, but hopefully this small review will be useful.

Complex analysis is concerned with the behavior of functions of complex variables. It turns out that these functions have a much di erent behavior than functions of real variables. The techniques of complex analysis, pioneered in the eighteenth and nineteenth centuries, have become very important in almost all branches of applied mathematics. I am assuming that you are familiar with the basics of complex numbers ($z = x + iy \ 2 \ C$) and will immediately move on.

In my experience, applications of complex numbers fall into two general catagories: Cauchy's Integral Theorem and Conformal Mapping. This review will consider the calculus of complex variables. I will not mention too much about conformal mapping, except to de ne it and give an example or two.

Complex numbers are written as z. Then a function of a complex variable has the form:

$$f(z) = f(x + iy) = u(x; y) + iv(x; y)$$

where $u; v$ are both real valued functions of the real variables $x; y$. You can also write a complex number in polar coordinates $z = re^{i}$, where $r \ j \ zj$ and is called the argment of z, sometimes denoted as arg z.

6.1 Analytic Functions

De nition: A complex function $f(z)$ is analytic at a point z_0 if it has a Taylor expansion that converges in a neighborhood of that point:

$$f(z) = \sum_{i=0}^{1} a_i (z \ z_0)^i$$

If $f(z)$ is analytic at all points inside some domain $D \ C$ then it is said to be analytic in D. If it is analytic for all nite complex numbers, it is said to be entire . If it is analytic for all complex numbers, including in nity, then $f(z) =$constant!

Theorem: The real and imaginary parts of an analytic function $f(z) = u(x;y) + iv(x;y)$ satisfy the Cauchy-Riemann (CR) Equations :

$$u_x = v_y \tag{6.1.1}$$

$$v_x = u_y \tag{6.1.2}$$

where u_x means the partial derivative of u with respect to x, etc. From the CR equations, it can be seen that both u and v satisfy Laplace's equation:

$$u_{xx} + u_{yy} = v_{xx} +$$

As an example, let's look at our old example of $f(z) = z^2$. This function agrees with the function $f(x) = x^2$ in the domain of real numbers ($y = 0$). It is also analytic, therefore it is the analytic continuation of the real-valued function $f(x) = x^2$. Further, it is unique: there is no other function that is analytic and matches with our original function!

That was an obvious one. What about a more complicated example? Consider the factorial function de ned on N: $f(n) = n!$ I will now de ne a function on the real numbers:

$$(x) = \int_0^1 ds\, s^{x-1} e^{-s} \tag{6.1.3}$$

Integrating by parts shows that when x is a natural number, $(x) = (x-1)!$ Furthermore, (x) is in fact an analytic function of x, so long as x is not a negative integer or zero. Hense, (x) is the (unique) analytic continuation of the factorial function to the real numbers (without the negative integers or zero). Indeed, we can go further: by replacing $x \to z$ (as long as z is not a negative integer or zero { why not?) this extends to all of C.

6.2 Singularities and Residues

De nition: A Laurent Expansion of a function $f(z)$ about a point z_0 is an expansion in both positive and negative powers:

$$f(z) = \sum_{n=-1}^{\infty} c_n (z - z_0)^n = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{6.2.4}$$

As you can see, a Laurent expansion is a generalization of a Taylor expansion. Notice that the function is not de ned at the point z_0 unless all of the $b_n = 0$. If this is the case, $f(z)$ is analytic at z_0. A Taylor expansion typically exists in an open disk about the point z_0. The generalization to the Laurent expansion converges inside an annulus $R_1 < |z - z_0| < R_2$.

A singularity of a function is a point where the function is not de ned. If z_0 is a singularity of $f(z)$ and you can de ne a neighborhood (open set) about that point $U(z_0)$ such that $f(z)$ is analytic in the domain $U(z_0) - \{z_0\}$ (so $R_1 \to 0$ above), then z_0 is called an isolated singularity . There are three general types of isolated singularities:

1. Removable singularities are the most mild form of singularity, where all the $b_n = 0$. An example of this is the function $\frac{\sin x}{x}$. This function is not de ned ousbu00 ousit01o91odo-27(es)-24oushye a wi27(e)ll-e ned alimi00 ous aY81(uu)-241(aan)-24

2. If a function has a Laurent expansion about z_0 but there exists an integer $m > 0$ such that all the $b_n = 0$ for every $n > m$ (but with $b_m \neq 0$), then the function is said to have a Pole of order m at z_0. In the special case of $m = 1$, the function is said to have a simple pole . An example of a function with a simple pole is $\frac{\cos x}{x}$ at the origin, where it behaves as $1=x$. Poles are very useful and important in complex analysis, as we will see.
3. If a function has a Laurent expansion about z_0 but there is not an integer $m > 0$ such that all the $b_n = 0$ for every $n > m$ (so the denominator terms do not stop), then the function is said to have an Essential Singularity at z_0. An example of this is the function $f(z) = e^{1=z}$ at the origin. This singularity is a true nightmare, and there is very little to be done for it!

If a function $f(z)$ has a pole of order m at z_0 (not necessarely a simple pole), then the quantity b_1 is called the Residue of the pole, also denoted by $\mathrm{Res}_{z=z_0} f(z)$. This quantity plays a vital role in the calculus of complex-valued functions. A function that is analytic in a domain, up to maybe a countable number of poles (so it still has a Laurent expansion), is said to be meromorphic in that domain.

Residues are so important, I want to list various methods of nding them. I will state several methods here without proof:

1. If $p(z_0) \neq 0$ and $q(z)$ has a zero of order m at z_0[1] then $\frac{p(z)}{q(z)}$ has a pole of order m at z_0.
2. If $f(z)$ has a pole of order m at z_0, it can be written as $f(z) = \frac{p(z)}{(z\ z_0)^m}$ where $p(z_0) \neq 0$. Then the residue of $f(z)$ at z_0 is $b_1 = {}^{p(m}$

Branch cuts are lines in the complex plane that tell you where to cut o the de nition of your function. For example, the branch cut for the logarithmic function is a line that connects the origin to in nity. For the principle branch, that line extends along the negative real axis. It could have gone in any direction, which would have corresponded

$$\int_C f(z)dz \qquad \int_a^b f(z(t))z'(t)dt \tag{6.3.6}$$

Cauchy-Goursat Theorem Let C be a closed contour in the complex plane and f (z) an analytic function on the domain interior to and on C. Then:

$$\oint_C f(z)dz \quad 0 \tag{6.3.7}$$

This is a generalization of the fundamental theorem of calculus. A powerful consequence of this theorem is that a contour integral is invariant to deformations of the contour, as long as the deformation does not pass through a singularity of f (z).

Cauchy's Integral Formula: Let f (z) be analytic everywhere within and on a simple, closed, positively oriented contour[2] C, and let z_0 be any point on the interior to C. Then:

$$f(z_0) = \frac{1}{2\,i} \oint_C \frac{f(z)dz}{z \quad z_0} \tag{6.3.8}$$

Using the Cauchy integral formula and some complex analysis, we can actually derive a very powerful formula for the n-th derivative of an analytic function:

$$f^{(n)}(z) = \frac{n!}{2\,i} \oint_C \frac{f(z)dz}{(z \quad z_0)^{n+1}} \tag{6.3.9}$$

From this, we come to a rather amazing result: if a function is analytic (which is the same as saying $\frac{df}{dz}$ exists), then all of its derivatives exist and are given by the above integral! This is very di erent from real analysis, where a function that has a rst derivative need not have higher derivatives. For example, $x^{3=2}$ has a rst derivative at $x = 0$ but it does not have a second derivative there. This kind of thing can never happen in complex analysis. Exercise: why doesn't $z^{3=2}$ contradict this result?

Notice immediately the marvelous corollary to the above formulas: for all n 2 Z,

$$\oint_C \frac{dz}{(z \quad z_0)^n} = 2\,i \quad _{n1} \tag{6.3.10}$$

as long as C encloses z_0. This follows for $n > 1$ sin

Using this result, we come to a remarkable and vital theorem used throughout all areas of applied and abstract mathematics. Consider a general, meromorphic function on the complex plane. Such a function has a Laurent expansion. Now let's do a closed contour integral, where we make sure the contour does not pass through any singularities of f (z). Some of the isolated singularities of f (z) are on the interior of C (call this set $Z = f z_0; z_1; : : : ; z_n g$) and some are on the exterior, where Z can be a nite or in nite set. Then we have the amazing and most important result:

$$\oint_C f(z)dz$$

$$\lim_{R!1} I_+ = \frac{e^{\ a}}{16a^5}(3 + a(a + 3))$$

To compute I

$$\boxed{\lim_{\epsilon \to 0} \int_{-1}^{x_0 - i\epsilon} \frac{f(x)}{x - x_0}\,dx = P\int_{-1}^{1} \frac{f(x)}{x - x_0}\,dx \quad \text{if } f(x_0)} \tag{6.3.13}$$

where the principle value on the right hand side should be understood as

$$P\int_{-1}^{1} \frac{f(x)}{x - x_0}\,dx \equiv \lim_{R \to \infty} \lim_{\epsilon \to 0} \left[\int_{-R}^{x_0 - \epsilon} \frac{f(x)}{x - x_0}\,dx + \int_{x_0 + \epsilon}^{R} \frac{f(x)}{x - x_0}\,dx \right] \tag{6.3.14}$$

This theorem goes by many names, but I like to call it the Principle Value Theorem ,

As a nal example of how complex calculus is so very useful, let's consider how to integrate a function with a branch-point singularity. This is a little di erent since a branch point is not an isolated singularity, so the residue theorem doesn't help by itself. But we can still do pretty well.

When there is a branch-point singularity, you have to specify a branch cut that connects the singularities. So for example, let's consider the following integral:

$$\int_0^1 \frac{x^a dx}{x+1}$$

where $0 < a < 1$. This integrand is multiply de ned, so we must choose a branch cut to de ne it. This is a line connecting the two branch points, in this case the origin and in nity. This comes from the fact that we can write the numerator as $e^{a \log x}$. Also, notice the simple pole at $x = 1$, so we shouldn't choose our branch cut to overlap with that pole. We will make our cut along the positive real axis. This integral is the same as

$$\int_{R_+} \frac{z^a dz}{z+1}$$

where the integral is along the positive real axis in the complex plane.

We will extend this contour: rst, the contour goes from $+i$ (< 1) to $R + i$ ($R > 1$) on the real axis (L_+). Then the contour goes counterclockwise around a circle of radius R until we get back to just below the real axis $R \; i$ (C_R). Then we travel back down the real axis until we get to $\; i$ (L

$$\int_{L_+} \frac{z^a dz}{z+1} = \int_R \frac{r^a dr}{r+1} \qquad \int_L \frac{z^a dz}{z+1} = \int_R \frac{r^a e^{2ia}\, dr}{r+1}$$

Putting all of this together and taking the limits $\to 0$ and $R \to 1$ gives

$$\int_0^1 \frac{r^a dr}{r+1} = \frac{2\pi i e^{ia}}{1 - e^{2ia}} = \frac{\pi}{\sin a}$$

6.4 Conformal Mapping

Conformal mapping is one of those tools that, in my experience, takes a great deal of

$$\arg w_1^0(t) \quad \arg w_2^0(t) = \arg z_1^0(t) \quad \arg z_2^0(t) \tag{6.4.16}$$

So indeed we nd that conformal transformations do preserve angles between contours. Before going further, let me point out a very important fact without proof: let f

Consider the mapping:

$$w = i\frac{1 - z}{1 + z} \tag{6.4.17}$$

This is an example of a Möbius Transformation, also known as a linear fractional transformation. It has the effect of mapping the interior of the unit circle in C_z to the entire upper-half plane of C_w, where the upper half of the circle is mapped to the positive real w axis while the lower half of the circle is mapped to the negative real axis. I leave it to you to prove that this map is, indeed, a conformal map.

Now the next step is to find a map on C_w that is harmonic and satisfies the boundary conditions. Consider the function

$$\frac{1}{\pi}\log w = \frac{1}{\pi}(\log|w| + i\arg(w)) \tag{6.4.18}$$

which is analytic and spans the upper half plane of C_w for $0 \leq \arg w \leq \pi$. Notice that the imaginary part of this function satisfies the boundary conditions we need. Since it is the imaginary component of an analytic function is harmonic, so we found our solution.

By writing $w =$

etc. When you've done problem after problem, you begin to develop an intuition for finding the right function. But it is something of an art, and like all forms of art, it takes practice to get good at it. But the rewards are worth it! Once you get good at finding the conformal map and the analytic function, you have completely solved the problem. You even get the field flux lines for free.

Suggested Reading

Here I include some books that I used when writing this review. It is by no means an exhaustive list, but it might be enough to get you started if you wish to follow up on any topic.

Brown, J. W. and Churchill, R. V., Complex Variables and Applications (6th Ed), McGraw Hill, Inc. (New York), 1996. This is a great book on complex analysis and conformal mapping. I heartily recommend it.

Dirac, P. A. M., General Theory of Relativity, Princeton University Press (New Jersey), 1996. This pamphlet has many formulas, but no exercises or details. A great place to see everything without any excess baggage, but I wouldn't use it as a primary source.

discussion of Lagrangian field theory, Noether's theorem, as well as a section on Lie groups.

Sakurai, J. J., Modern Quantum Mechanics, Addison-Wesley (Massachusetts), 1994. A textbook used for graduate quantum mechanics in many universities. It has a very good discussion of the theory of angular momentum, where tensors and Lie groups have a secure home in physics.

Ward, R. M., General Relativity, University of Chicago Press (Illinois), 1984. A great book { one of my favorites. It has many details and is accessible to graduates.

I hope that this review is of use to you, and that these references can help you to fill in the gaps, or at least start you on your way. If you have any suggestions for the improvement of this paper, please send me your comments (blechman@pha.jhu.edu).