The Free Klein Gordon Field Theory

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Abstract

A single-particle relativistic theory turns out to be inadequate for many situations. Thus, we begin to develop a multi-particle relativistic description of quantum mechanics starting from classical analogies. We start with a Lagrangian description, and use it to build a Hamiltonian description we can then quantize. We then illustrate the decomposition of the field into its positive and negative energy components, and define the creation and annihilation operators.

1 Introduction

Relativistic quantum mechanics turns out to be inadequately described by a single-particle theory. One of the primary reasons for this is that a relativistic particle may have enough energy to create other particles, and thus any theory describing it must account for this.

A theory which describes an infinite-degree-of-freedom system could be built out of a function defined over allersystebuitoile5205tow8(era7(y)-4408(bsuc7(y)hTJ 0 -11.955 Td [(pais.ng-44

We will assume there exists an action S defined as normal, which we can write in terms of $\mathcal{L}$:

$$S = \int_{t_i}^{t_f} dt\, L = \int_{t_i}^{t_f} dt\, d^3x\, \mathcal{L} = \int_{t_i}^{t_f} d^4x\, \mathcal{L} \qquad (2)$$

We will further assume that $\mathcal{L}$ depends only on the field variable and its first derivative, i.e.

$$\mathcal{L} = \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \qquad (3)$$

Now we will change the field arbitrarily and infinitesimally to find what conditions will result in a stationary action

$$\phi(x) \rightarrow \phi'(x) = \phi(x) + \delta\phi(x) \qquad (4)$$

and apply the vanishing boundary condition

$$\delta\phi(\mathbf{x}; t_i) = \delta\phi(\mathbf{x}; t_f) = 0 \qquad (5)$$

Now we can look at an infinitesimal change in S:

$$\delta S = \int_{t_i}^{t_f} d^4x\, \delta\mathcal{L} \qquad (6)$$

We can expand this using the total derivative of $\mathcal{L}$:

$$= \int_{t_i}^{t_f} d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi(x)} \delta\phi(x) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right.$$

We will pick the Lagrangian density

$$L = \frac{1}{2}\partial \quad \partial \qquad \frac{1}{2}m^2 \quad ^2 \tag{11}$$

This choice can be motivated by looking at the Lagrangian for a classical harmonic oscillator,

$$L = \frac{1}{2}mv^2 \quad \frac{1}{2}m\omega^2x^2 \tag{12}$$

We see that if we treat $\ (x)$ as x, $\partial \quad$ is analogous to v, and doing so gives us (11).

Evaluating both sides to be

$$\frac{\partial L}{\partial \ (x)} = \quad m^2 \ (x) \tag{13a}$$

$$\frac{\partial L}{\partial \partial \quad (x)} = \frac{\partial}{\partial \partial} \quad \frac{1}{2}\partial \qquad \partial \tag{13b}$$

$$= \frac{1}{2} \qquad \frac{\partial}{\partial \partial}\partial \qquad \partial \quad + \partial \qquad \frac{\partial}{\partial \partial}\partial \tag{13c}$$

$$= \frac{1}{2} \quad (\partial \quad + \partial \qquad) \tag{13d}$$

$$= \frac{1}{2} \quad (2\partial \quad) \tag{13e}$$

$$= \partial \tag{13f}$$

with the Hamiltonian itself defined as

$$H = \int d^3x\, \mathcal{H} \tag{17}$$

Clearly, these are defined very similarly to their classical definitions.

equations with :

$$
\begin{aligned}
\dot{}(x) &= f\ (x); H g \\
&= \ (x);\ \int d^3y \ \ \frac{1}{2}\ {}^2(y) + \frac{1}{2}(r\ (y))\ (r\ (y)) + \frac{1}{2}m^2\ {}^2 \\
&= \frac{1}{2}\int d^3x\, f\ (x);\ {}^2(y) g_{x^0=y^0} \\
&= \int d^3y\ \ (y) f\ (x);\ {}^2(y) g_{x^0=y^0} \\
&= \int d^3y\ \ (\mathrm{y}; x^0) f\ (x);\ (y) g_{x^0=y^0} \\
&= \int d^3y\ \ (\mathrm{y}; x^0)\ {}^3(x\ \ y) \\
&= \ (x) \qquad (25)
\end{aligned}
$$

Here we have used that the Hamiltonian is time-independent, so we can assert that $x^0 = y^0$, allowing us to use the equal-time relation (19). We can go through a similar process to nd $\dot{}$:

$$
\begin{aligned}
\dot{}(x) &= f\ (x); H g \\
&= \ (x);\ \int d^3y \ \ \frac{1}{2}\ {}^2(y) + \frac{1}{2}(r\ (y))\ (r\ (y)) + \frac{1}{2}m^2\ {}^2 \\
&= \int d^3y\,(r\ (y)\ \ f\ (x); r\ (y) g_{x^0=y^0} \\
&\qquad + m^2\ (y) f\ (x);\ {}^2(y) g_{x^0=y^0}) \\
&= \int d^3y\,(r\ (\mathrm{y}; x^0)\ \ r(\ \ {}^3(x\ \ y)) \\
&\qquad + m^2\ (\mathrm{y}; x^0)(\ \ {}^3(x\ \ y))) \\
&= r\ \ r\ (x)\ \ m^2\ (x) = r^2\ (x)\ \ m^2\ (x) \qquad (26)
\end{aligned}
$$

These rst-order Hamilton equations give us the second-order equation

$$
\begin{aligned}
\ddot{}(x) = \dot{}(x) = r^2\ (x)\ \ m^2\ (x) \\
!\ (\partial^2\ \ r^2\ (x)) + m^2\ (x) \text{ equations}
\end{aligned}
$$

4 Field Decomposition

The following plane wave equation set forms a complete basis for solutions to the Klein-Gordon equation [1]:

$$(x) = e^{\ ik\ x} \tag{30}$$

We can use this basis to expand in this basis:

$$(x) = C \int d^4k\, e^{\ ik\ x}\,\tilde{}(k) \quad ; \quad C = \frac{1}{(2\ )^{\frac{3}{2}}} \tag{31}$$

This is essentially a Fourier transform of $\tilde{}(k)$, with C introduced for later convenience. We can now plug this into the Klein-Gordon equation:

$$\begin{aligned}
& C(\partial\ \partial\ \ + m^2) \int d^4k\, e^{\ ik\ x}\,\tilde{}(k) = 0 \\
& \rightarrow C \int d^4k\,(\partial\ \partial\ \ + m^2)e^{\ ik\ x}\,\tilde{}(k) = 0 \\
& \rightarrow C \int d^4k\,(\ k^2 + m^2)\,\tilde{}(k)e^{\ ik\ x} = 0
\end{aligned} \tag{32}$$

So at least one of the following must be true:

$$k^2 = m^2 \tag{33}$$

$$\tilde{}(k) = 0 \tag{34}$$

So, $\tilde{}$ is only non-vanishing when $k^2 = m^2$; we'll de ne $\tilde{}$ in the following way to capture this:

$$\tilde{}(k) = \text{So, } \tilde{}$$

Now we can plug this new form into (35b):

$$(x) = C \int dk^0\, d^3k\, \frac{1}{2E_k} (\ (k^0 \quad E_k) + \ (k^0 + E_k))$$
$$e^{\quad i(k^0 x^0 \quad \mathbf{k}\ \mathbf{x})} a(k^0; \mathrm{k})$$
$$= C \int d^3k\, \frac{1}{2E_k} \quad e^{i(\quad E_k x^0 + \mathbf{k}\ \mathbf{x})} a(E_k; \mathrm{k}) + e^{i(E_k x^0 + \mathbf{k}\ \mathbf{x})} a(\quad E_k; \mathrm{k}) \qquad (39)$$

We will replace E_k with k^0 and switch **k** to **k** in the second term to obtain

$$(x) = C \int d^3k\, \frac{1}{2E_k} \quad {}^{x)}$$

References

[1] Ashok Das. *Lectures on Quantum Field Theory*. World Scienti c, 2008.

[2] Ben Saltzman. Second quantization of the klein-gordon equation. 2018.