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QUANTUM FIELD THEORY II

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Contents

1 Introduction

2 The Dirac Equation

2.1 Derivation From Scratch

The Dirac Equation has to be relativistic, and so a logical place to start our derivation is equation (1). If you're wondering where equation (1) comes from, it's quite simple. When you think of physics, one of the first equations that comes to mind is the incredibly famous

$$E = mc^2 \tag{4}$$

This equation gives the energy of a particle of mass m at rest. If the particle is moving with momentum $|\vec{p}|$, this equation becomes the more general

$$E^2$$

We want to nd a way to make equation (11) coherent. To do this, we need to transform the third term into a scalar. An obvious way is to make the following transformations

$$\begin{aligned} p \quad ! \quad p \\ p \quad ! \quad p \end{aligned} \tag{12}$$

Thus,

$$(p \quad m)(p \; + m) \; ! \; (\quad p \quad m)(\quad p \; + m) = 0 \tag{13}$$

which must equal to $p \; p \quad m^2$.

Multiplying terms, we get the following two equations

$$\begin{aligned} p \; = \; p \\ p \quad p \; = p \; p \end{aligned} \tag{14}$$

The rst equation leads to = = . Plugging this in the second equation of (14), using the identity $A \; B \; = A \; B$, and noting that p and act on di erent spaces and therefore commute, we get

$$p$$

[(satisfy) -279 (the) -2710.364.339 Td =5

2.2 A Deeper Look at the Gamma Matrices

We found that the Dirac matrices satisfy the Cli ord Algebra given by the ab1f27(y)-279()-278()27(y)-2-2

2.3 Pauli's Fundamental Theorem

Pauli's Fundamental Theorem states that if $[\quad ; \quad]_{+} = 2$ and $[\ ^{0} ; \ ^{0}]_{+} = 2$, then there exists a constant invertible matrix S such that

$$^{0} = S \quad S^{1} \tag{28}$$

This isn't hard to believe from a linear algebra perspective since the are Hermitian. There are multiple representations of the matrices. We want to show here that changing representation should not a ect the underlying physics, it should be like changing your frame of reference. Let's nd a relation between the Dirac Equation for these two representations. Starting with the prime representation, we have

$$\begin{aligned}
(\ ^{0} p \quad m)\ ^{0} = 0 \ &, \ (S \quad S^{1} p \quad m)\ ^{0} = 0 \\
&, \ S(\quad p \quad m) S^{1}\ ^{0} = 0 \\
&, \ (\quad p \quad m) S^{1}\ ^{0} = 0 \\
&, \ (\quad p \quad m) \quad = 0
\end{aligned} \tag{29}$$

Thus, the two equation are equivalent given the relation $^{0} = S$, which obviously doesn't change the physics since it corresponds to a rotation.

2.4 The Hamiltonian of the Dirac Equation

We want to express the Dirac Equation in the familiar form $H \quad = E \quad = i@_0$. But

$$\begin{aligned}
(i \quad @ \quad m) \quad = (i\ ^{0}@_0 + i\ ^{i}@_i \quad m) \quad &= 0 \\
, \ (i\ ^{i}@_i \quad m) \quad &= \quad i\ ^{0}@_0 \\
, \ (i\ ^{0}\ ^{i}@_i \quad m\ ^{0}) \quad &= \quad i^{0}@_0 \\
, \ (\quad i\ ^{0}\ ^{i}@_i + m\ ^{0}) \quad &= i^{0}@_0 \quad = E
\end{aligned} \tag{30}$$

Thus,

$$\begin{aligned}
H = \quad i\ ^{0}\ ^{i}@_i + m\ ^{0} &= \ ^{0}\tilde{} \ (\ i\, r\,) + m\ ^{0} \\
&= \ ^{0}\tilde{} \ \not{p} + m\ ^{0}
\end{aligned} \tag{31}$$

Letting

$$\tilde{} = \ ^{0}\tilde{} \tag{32}$$

and

$$= \ ^{0} \tag{33}$$

so

$$\tilde{} = \begin{matrix} 0 & \tilde{} \\ \tilde{} & 0 \end{matrix} \tag{34}$$

and

$$= \begin{matrix} I & 0 \\ 0 & I \end{matrix} \tag{35}$$

Then, the Hamiltonian becomes

$$\boxed{H = \tilde{} \quad \not{p} + m}$$

The Hamiltonian is an observable and therefore has Tf 8.0a.936 Td [(p)]TJ/F38 7.9701 Tflc0.398hnd oar

$$\begin{aligned}(\vec{\alpha}\cdot\vec{p})^\dagger = \vec{\alpha}\cdot\vec{p} &\to \vec{\alpha}^\dagger = \vec{\alpha} \\ \beta^\dagger = \beta &\to (\gamma^0)^\dagger = \gamma^0\end{aligned} \tag{36}$$

The first equation leads to

$$\begin{aligned}(\gamma^0\vec{\gamma})^\dagger &= \gamma^0\vec{\gamma} \\ \Leftrightarrow (\vec{\gamma})^\dagger\gamma^0 &= \gamma^0\vec{\gamma} \\ \Leftrightarrow (\vec{\gamma})^\dagger &= \gamma^0\vec{\gamma}\gamma^0 = -\vec{\gamma}\end{aligned} \tag{37}$$

where in the last step we've used (19) and (21). To summarize, we just derived the following

$$\begin{aligned}(\gamma^0)^\dagger &= \gamma^0 \\ (\gamma^i)^\dagger &= -\gamma^i\end{aligned} \tag{38}$$

3 Solutions of the Dirac Equation

3.1 Spin of the Dirac Particle

What type of particle is a Dirac particle? Note that rotations should be a symmetry of the system, and therefore the total angular momentum $\vec{J}$ must be a conserved quantity (i.e., constant in time). From

$$\frac{d\vec{J}}{dt} = i\left[H, \vec{J}\right] \tag{39}$$

we must have

$$\left[H, \vec{J}\right] = 0$$

From Quantum Mechanics, we normally have $\vec{J} = \vec{L} + \vec{S}$, but maybe in this case we just have $\vec{J} = \vec{L}$. Let's try this

$$\begin{aligned}H = \vec{\alpha}\cdot\vec{p} + m\beta &= \alpha_n p_n + m\beta \\ \vec{L} = \vec{x}\times\vec{p}\ , \quad L_i &= \epsilon_{ijk}x_j p_k\end{aligned} \tag{40}$$

$$\begin{aligned}[L_i, H] = [\epsilon_{ijk}x_j p_k, \alpha_n p_n + m\beta] &= \epsilon_{ijk}\alpha_n[x_j, p_n]p_k \\ &= i\delta_{jn}\epsilon_{ijk}\alpha_n p_k \\ &= i\epsilon_{ijk}\alpha_j p_k \neq 0\end{aligned} \tag{41}$$

Therefore, we must have $\vec{J} = \vec{L} + \vec{S}$, and

$$[S_i, H] = -i\epsilon_{ijk}\alpha_j p_k \tag{42}$$

What S_i would satisfy that? We know from Quantum Mechanics that $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$, which is pretty close, so we probably want to play with the α_n in the Hamiltonian.

Suppose S_i only interacts with $\vec{\alpha}$, then

$$\begin{aligned}[S_i, H] = [S_i, \alpha_k p_k + m\beta] &= [S_i, \alpha_k]p_k \\ &= -i\epsilon_{ijk}\alpha_j p_k\end{aligned} \tag{43}$$

Thus, if we can find an S_i such that $[S_i, \alpha_k] = -i\epsilon_{ijk}\alpha_j$ we are done. This is very close to $[\sigma_i, \sigma_j]$. To summarize, we want to get

$$[S_i, \alpha_j] = -i\epsilon_{ikj}\alpha_k = i\epsilon_{ijk}\alpha_k \tag{44}$$

Considering the σ dependence of α, a good guess is $S_i = \sigma_i$. Then,

$$[\ \ _i;\ \ _j] = \begin{matrix}[\ \ _i;\ \ _j] & 0 \\ 0 & [\ \ _i;\ \ _j]\end{matrix} = 2i\ \ _{ijk} \begin{matrix} \ _k & 0 \\ 0 & \ _k \end{matrix} \neq i\ \ _{ijk}\ \ _k \tag{45}$$

So this doesn't work. Another good guess would be

$$S_i = \vec{\ }_i = \begin{matrix} \ _i & 0 \\ 0 & \ _i \end{matrix} \tag{46}$$

Then

$$[\vec{\ }_i;\ \ _j] = \begin{matrix} 0 & [\ \ _i;\ \ _j] \\ [\ \ _i;\ \ _j] & 0 \end{matrix} = 2i\ \ _{ijk} \begin{matrix} 0 & \ _k \\ \ _k & 0 \end{matrix} = 2i\ \ _{ijk}\ \ _k \tag{47}$$

Therefore, letting $S_i = \frac{1}{2}\vec{\ }_i$

From the ? part of the derivation, we can see the doubly degenerate behavior of the solutions. That should ring a bell since we just saw this for the spin. To underline this property, we want to split $u(p$

and

$$\chi(p) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{66}$$

The rst choice of $\chi(p)$ leads to

$$u(p) = \begin{pmatrix} \frac{p_3}{E \quad m} \\ 0 \end{pmatrix} \tag{67}$$

and the second choice of $\chi(p)$ leads to

$$u(p) = \begin{pmatrix} 0 \\ \frac{p_3}{E \quad m} \end{pmatrix} \tag{68}$$

Finally, we have our four solutions!

4 Continuity Equation

The big problem with the Klein-Gordon Equation was the possible negative probability density. We got rid of the problem by working in analogy to the Schroedinger Equation and requiring rst order in time. Hopefully this works out, otherwise everything we just accomplished is a bit useless... Let's see if it does end up working out (hopefully). From $H \quad = E \quad$, which we found before to be equivalent to

$$(\quad i\vec{\ } \quad \nabla + \quad m) \quad = i\frac{\partial}{\partial t} \tag{69}$$

we need to get something of the form

$$\frac{\partial}{\partial t} + \nabla \quad \vec{j} = 0 \tag{70}$$

which if you took some electromagnetism should look familiar. It is called the continuity equation. If we can nd a way to remove the $\quad m$ term, we would be good. Recalling that we must have $H = H^{\dagger}$ and so $\vec{\ } = \vec{\ }^{\dagger}$ and $\quad = \quad ^{\dagger}$, we get

$$^{\dagger}(i \nabla \quad \vec{\ } + \quad m) = \quad i\frac{\partial}{} \quad ^{\dagger}$$

where we have used the identity $\nabla \cdot (f\vec{A}) = (\nabla f) \cdot \vec{A} + f(\nabla \cdot \vec{A})$. It follows that letting $\rho = \psi^\dagger \psi = |\psi|^2 \geq 0$ and $\vec{j} = \psi^\dagger \vec{\alpha} \psi$, we are done. Good! We do indeed have the required positive probability density $\rho \geq 0$, so our previous work is relevant to the real world.

A more concise way to write the above continuity equation can be found by defining the following four vector current

$$j^\mu = (\rho, \vec{j}) = (\psi^\dagger \psi, \psi^\dagger \vec{\alpha} \psi) \tag{74}$$

then the continuity equation is equivalent to

$$\partial_\mu j^\mu = 0 \tag{75}$$

5 Conclusion

> *The mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by Nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which Nature has chosen.*
>
> Paul Dirac (1902-1984)

In this lecture, we have looked at one of the most beautiful equation in physics: the Dirac Equation. We first figured out why we need it, then from these arguments we derived it from scratch. We then discovered some of its amazing properties like how the Dirac gamma matrices must behave, what its Hamiltonian must be like, the necessity of the particles it describes having spin. Finally, we put everything together to understand what the plane wave solutions for these particles describes and if the probability density is strictly positive, which we need in order to not have negative probabilities popping up.

References

[1] Ashok Das, *Lectures on Quantum Field Theory*, World Scientific, 2008, First Edition, pp. 19-47.