1 Introduction

The Schrödinger equation, successful as it is for describing non-relativistic quantum particles, fails in the relativistic regime. By using special relativity, one can derive a relativistic version of the Schrödinger equation, known as the first quantization of the Klein-Gordon equation. Unfortunately, this has its own problems: negative energy and probability solutions, nonphysical entities that could spell the end for a theory. Moving from quantum theory to quantum field theory, fields are introduced as fundamental. Quantizing the Klein-Gordon equation in quantum field theory leads to a model known as the second quantization, which avoids many of the problems of the first quantization.

2 Second Quantization of the Klein-Gordon Equation

2.1 The Klein-Gordon Field

We can write the Klein-Gordon field operator as

$$\varphi(x) = C \quad d^3k$$

Inverting equations 1 and 5 and solving for the annihilation and creation operators a and $a^{\dagger}$:

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$$[\varphi(\mathrm{x}),\varphi(\mathrm{y})] = CC \int\int \mathrm{d}^3k \; \mathrm{d}^3k' \Big\{ e^{-i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{y})}[a(\mathrm{k}),a(\mathrm{k}')]$$

$$+ e^{-i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{y})}\big[a(\mathrm{k}),a^\dagger(\mathrm{k}')\big] + e^{-i(-\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{y})}\big[a^\dagger(\mathrm{k}),a(\mathrm{k}')\big]$$

$$+ e^{-i(-\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{y})}\big[a^\dagger(\mathrm{k}),a^\dagger(\mathrm{k}')\big]\Big\}$$

$$= 0 \tag{8}$$

$$[\Pi(\mathrm{x}),\Pi(\mathrm{y})] = -k^0k'^0CC \int\int \mathrm{d}^3k \; \mathrm{d}^3k' \Big\{ e^{-i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{y})}[a(\mathrm{k}),a(\mathrm{k}')]$$

$$- e^{-i(\mathbf{k}$$

In the last step, we use the fact that $k^0 = k'^0$. Similarly, we find that

$$\left[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = 0 \tag{12}$$

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = \delta^3(\mathbf{k} - \mathbf{k}') \tag{13}$$

So the creation and annihilation operators commute with themselves for any two $\mathbf{k}$ and $\mathbf{k}'$, but do not commute with each other.

2.2 Relationship with the Harmonic Oscillator

The Hamiltonian is the integral of the Hamiltonian density

$$= \frac{1}{2} \int d^3k \; k^0 \left(a(\mathbf{k})a^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k})a(\mathbf{k}) \right)$$

$$= \frac{1}{2} \int d^3k \; E_k \left(a(\mathbf{k})a^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k})a(\mathbf{k}) \right) \tag{19}$$

Recalling the commutation relations from equations 2.1, 12, and 13, the where the number operator $N(\mathbf{k}) = a^{\dagger}(\mathbf{k})a(\mathbf{k})$. The total number operator is

$$N = \int d^3k N(\mathbf{k}) \tag{24}$$

The commutator relations follow easily:

$$\begin{aligned}
[a(\mathbf{k}), N(\mathbf{k}')] &= \left[a(\mathbf{k}), a^{\dagger}(\mathbf{k}')\right] a(\mathbf{k}) \\
&= \delta^3(\mathbf{k} - \mathbf{k}')a(\mathbf{k}) \qquad (25) \\
[a(\mathbf{k}), N] &= a(\mathbf{k}) \qquad (26) \\
\left[a^{\dagger}(\mathbf{k}), N(\mathbf{k}')\right] &= a^{\dagger}(\mathbf{k}) \left[a^{\dagger}(\mathbf{k}), a(\mathbf{k}')\right] \\
&= -\delta^3(\mathbf{k} - \mathbf{k}')a^{\dagger}(\mathbf{k}) \qquad (27) \\
\left[a^{\dagger}(\mathbf{k}), N\right]
\end{aligned}$$

We can construct any higher state in a similar way: with repeated applications of the creation operator. With normalization, these higher states can be formed by

$$|\Xi\rangle = \frac{(a^\dagger(\mathbf{k_1}))^{n_1}}{\sqrt{n_1!}} \frac{(a^\dagger(\mathbf{k_2}))^{n_2}}{\sqrt{n_2!}} \dots \frac{(a^\dagger(\mathbf{k_l}))^{n_l}}{\sqrt{n_l!}} |0\rangle \tag{32}$$

where there are n_i particles with momentum $\mathbf{k_i}$. It is clear that there should be $\sum_{i=1}^{l} n_i$ particles in this state, so

$$N|\Xi\rangle = (n_1 + n_2 + \cdots + n_l)|\Xi\rangle \tag{33}$$

Moreover, using the identity $[A, B^n] = \sum_{i=0}^{n-1} B^{n-i-1}[A, B]B^i$ and equation 13, we can see that there are indeed n_i particles with momentum $\mathbf{k_i}$, as we expect.

$$\begin{aligned}\left[N(\mathbf{k}), (a^\dagger(\mathbf{k_j}))^{n_j}\right] &= a^\dagger(\mathbf{k})\left[a(\mathbf{k}), (a^\dagger(\mathbf{k_j}))^{n_j}\right] \\ &= a^\dagger(\mathbf{k}) \sum_{i=0}^{n_j-1} (a^\dagger(\end{aligned}$$

We can also introduce a momentum operator

$$\mathbf{P} = \int d^3k \; \mathbf{k} N(\mathbf{k}) \tag{36}$$

The actions of the Hamiltonian and momentum operators on our state $|\Xi\rangle$ can be determined easily from equations 35:

$$\begin{aligned} H\,|\Xi\rangle &= \int d^3k E_k N(\mathbf{k})\,|\Xi\rangle \\ &= \sum_{i=1} n_i E_{k_i}\,|\Xi\rangle \end{aligned} \tag{37}$$

$$\begin{aligned} \mathbf{P}\,|\Xi\rangle &= \int d^3k \mathbf{k} N(\mathbf{k})\,|\Xi\rangle \\ &= \sum_{i=1} n_i \mathbf{k_i}\,|\Xi\rangle \end{aligned} \tag{38}$$

The energy and momentum of the state, the eigenvalues of the Hamiltonian and momentum operators, are simply the sum of the energies and momenta of all of the particles in the state.

Consider the state in equation 14, $|k\rangle$. This represents a one particle state with four-momentum $k^\mu = (E_k, \mathbf{k})$. Clearly,

$$\begin{aligned} N\,|k\rangle &= |k\rangle \\ H\,|k\rangle &= E_k\,|k\rangle \\ \mathbf{P}\,|k\rangle &= \mathbf{k}\,|k\rangle \end{aligned} \tag{39}$$

In a similar way, we can apply the field operator itself to the ground state:

$$|\varphi(x)\rangle = \varphi(x)\,|0\rangle = \varphi^{(-)}(x)\,|0\rangle \tag{40}$$

The $phi^{(+)}$ term is zero because it involves a and not $a^\dagger$ (equation 4). The projection onto $|\varphi\rangle$ onto $|k\rangle$ is $\langle k|\varphi(x)\rangle$. This projection is a solution to the Klein-Gordon equation, just like φ itself:

$$\begin{aligned} (\partial_\mu \partial^\mu + m^2)\,\langle k|\varphi(x)\rangle &= \langle k|(\partial_\mu\partial^\mu + m^2)\varphi(x)|0\rangle \\ &= 0\,\langle k|0\rangle \\ &= 0 \end{aligned} \tag{41}$$

So $k|\varphi(x)$