Cosmological Tensor Perturbations

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Abstract

Previously, we calculated the effect of scalar perturbations, Φ and Ψ, of the Friedmann-LemaitreMk F

$$\mathbb{H}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (2)$$

Note that $_{\mu\nu}$ is symmetric, traceless, and divergenceless. Since we are in the Fourier domain, divergenceless means $k^i{}_{ij} = k^j{}_{ij} = 0$. What we have done so far is the only thought we have to put into this process. We now compute the Ricci Tensor and the Ricci Scalar by plugging and chugging and throwing out terms of O(2) or greater.

2. Christoffel Symbols for Tensor Perturbations

We start with $\Gamma^0{}_{\alpha\beta}$. Since Christoffel symbols are merely derivatives of the metric it is easy to see that

$$\Gamma^0_{00} = \Gamma^0_{i0} = 0 \qquad (3)$$

since the metric is constant for those indices. All of the Christoffel symbols with two lower indices are

$$\Gamma^0_{ij} = \frac{1}{2} g_{ij,0} \qquad (4)$$

We have used the notation where a comma followed by an index in the subscript indicates a partial derivative with respect to that index. We can write the spatial components of the metric as

$$g_{ij} = a^2(\delta_{ij} + \mathbb{H}_{ij}) \qquad (5)$$

Thus

$$g_{ij,0} = 2Hg_{ij} + a^2\mathbb{H}_{ij,0} \qquad (6)$$

Note that $\boldsymbol{H}$ represents the Hubble rate and note the perturbation tensor. Substituting (6) into (4) gives

$$\Gamma^0_{ij} = Hg_{ij} + \frac{a^2\mathbb{H}_{ij,0}}{2} \qquad (7)$$

$$R_{00} = \frac{-3}{a}\frac{d^2a}{dt^2} \qquad (14)$$

A much more manageable equation. We have seen this equation before. We found it when we calculated the Ricci Tensor for scalar perturbations and even before when we analyzed the unperturbed metric. Thus, at first-order, tensor perturbations do not affect the time-time component of the Ricci Tensor.

Now we have one last laborious task, calculating the spatial components of the Ricci Tensor

$$R_{ij} = \Gamma^{\alpha}_{ij,\alpha} - \Gamma^{\alpha}_{i\alpha,j} + \Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{ij} - \Gamma^{\alpha}_{\beta j}\Gamma^{\beta}_{i\alpha} \qquad (15)$$

It turns out to be convenient to consider the first two terms together. If we separate the time and space components, we get

$$\Gamma^{\alpha}_{ij,\alpha} - \Gamma^{\alpha}_{i\alpha,j} = \Gamma^{0}_{ij,0} + \Gamma^{k}_{ij,k} - \Gamma^{k}_{ik,j} \qquad (16)$$

Note that the second term on the left hand side is zero when $\boldsymbol{\alpha} = 0$. On the right hand side, note that $\Gamma^0_{ij} = g_{ij,0}/2$. Thus the first term can be written as $g_{ij,00}/2$. From (10) we can see that the last term vanishes. Thus, we end up with

$$\Gamma^{\alpha}_{ij,\alpha} - \Gamma^{\alpha}_{i\alpha,j} = \frac{g_{ij,00}}{2} + \frac{1}{2}[-k_i k_k \mathbb{H}_{jk} - k_j k_k \mathbb{H}_{ik} + k^2 \mathbb{H}_{ji}] \qquad (17)$$

We now can take advantage of coordinate choice and note that ***k*** lies in the ***z***-direction. This means that anywhere we see an index on ***k***, we replace it with 3. This kills the first two terms in the bracket leaving

$$\Gamma^{\alpha}_{ij,\alpha} - \Gamma^{\alpha}_{i\alpha,j} = \frac{g_{ij,00}}{2} + \frac{k^2}{2}\mathbb{H}_{ji} \qquad (18)$$

This is much more manageable but remember that it is only two terms in the Ricci tensor, we still have two more to calculate. We expand out the third term in (15)

$$\Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{ij} = \Gamma^{k}_{k0}\Gamma^{0}_{ij} + \Gamma^{k}_{kl}\Gamma^{l}_{ij} \qquad (19)$$

In the second term the Christoffel symbols are first-order so their product is second order and is thus neglected. For the first Christoffel symbol in the first term we look at (9), set $\boldsymbol{i = j}$, and sum. The first term in (9) becomes $3H$ and the second term is first-order so its product with the other part of the term in (19) is neglected. Using (7) we get

(20)

Using similar methods, we compute the final term in (15)

(21)

Putting it all together we get

(22)

We are not quite done yet. We must substitute in the time derivatives of the metric. First we evaluate the second time derivative using (6)

(23)

We then substitute (23) into (22) and we get

(24)

It is easy to compute the Ricci Scalar

(25)

Looking at the first-order part, we can ignore the first term which is zero-order. In the second term, contracting the metric gives zero-order terms for the terms in (24) proportional to the metric. Since all the other terms in (24) are first-order in the tensor perturbation, we take the zero-order part of the metric, $g^{jk} = \delta^{jk}/a^2$. This takes the tracter

With the Ricci Tensor and the Ricci Scalar, it is relatively easy to calculate the Einstein Tensor. You can then show that tensor perturbations give rise to gravitational waves. You can also see that the tensor perturbations are decoupled from the scalar perturbations.

This derivation required a lot of menial algebra and neglecting higher order terms. This is useful the first time you see such a problem to learn the techniques of the computation, but becomes tedious math if you want to see higher order terms or choose a new metric. Since this section was mostly a mathematics exercise with almost no physics, it makes sense to enlist the help of computers to do the tedious algebra of calculating the Christoffel Symbols, Ricci Tensor and Scalar, and the Einstein Tensor. I am currently working on doing symbolically this with Wolfram Mathematica. Hopefully, users will be able to input any metric and have Einstein's equations pop out. This would allow users to focus on the physics and leave the tedious algebra to the computers.

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