The Building Blocks for General Relativity in a Smooth, Expanding Universe

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Introduction

In order to properly study cosmology, a set of mathematical tools are required for describing the universe. Without these, one cannot evaluate the mathematical aspect of spacetime. There would be immense amounts of hand-waving and jumping to conclusions, as well as blind trust in the author and conceptual intuition. This is not physics. So, in order to be able to rigorously study and understand the universe, one must begin with the basics. In this case, basic meaning fundamental rather than simple. Here, a solid foundation is built from which more complex observations can be derived. The foundation in this case is the mathematical framework of general relativity.

Assumptions

A part of building the foundation is first evaluating the base case. Here, that means that it will be assumed that:

1. The universe is smooth, meaning no densities vary as a function of space.
2. The universe is in equilibrium.

Again, these things are not universally true

A helpful aspect of the metric is that it accounts for the effects of gravity on spacetime. Rather than considering gravity as an external force that affects matter, the metric allows for gravity to be built into the geometry of space.

Spacetime, however, is not two-0 G[(two)]TJETQq0.0000912 0 612 792 reW*nBT/F2 12 TfTJETQq0.0000912 0 6

In order to generalize this, the base case is once again a freely moving particle in two-dimensional Euclidian space, which yields:

Applying a known coordinate system once again, how can this be generalized to polar coordinates? The basis vectors for polar coordinates are and . These vectors vary over space, whereas the basis vectors for cartesian coordinates do not. Setting $x^i=(r,)$, the above condition does not imply the following condition:

Instead, the cartesian condition needs to be transformed. This is done using the transformation matrix as follows:

Thus transforming the velocity of the particle from cartesian coordinates to polar coordinates. The geodesic equation then becomes the following:

Note that the time derivatives cannot simply cancel out, as the transformation being used is not linear. The following equality emerges from the above equations:

And thus, the following geodesic equation arises:

The transformation is being used on the first term of the middle portion of the equality. Thus, this equality can be multiplied by the inverse of the transformation matrix to get a more palatable form.

From this equation, the Christoffel symbol can be defined as the term in the square brackets. This symbol becomes useful in many calculations withing general relativity.

To generalize the derived geodesic equation to general relativity, the range of the indices changes from 1 and 2 to 0 to 3.