Helicity, chirality, and the Dirac equation in the non-relativistic limit

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Abstract

The Dirac equation describes spin-1/2 particles with a consideration for the effects of special relativity. In this paper, we explore two major emergent results of the Dirac equation. First, we see how the notions of helicity and chirality arise from the Dirac equation, and exactly correspond to one another in the massless limit. Second, we verify that the Dirac equation is consistent with the Schrödinger equation in the non-relativistic limit, both for a free particle and for a charged particle in an external magnetic field.

1 Introduction

The Dirac equation, named after Paul Dirac, represented an attempt to incorporate the effects of special relativity into quantum mechanics, and was introduced in 1928 [2]. The result was a wave equation describing the relativistic behavior of spin-1/2 particles, such as electrons and neutrinos. A major departure from previous quantum theories, Dirac's equation describes particles

particles of a given chiral handedness express helicity of the same handedness.

2.1 The helicity operator

The Dirac Hamiltonian

$$H = \alpha \cdot \mathbf{p} + \beta m \tag{1}$$

does not necessarily commute with orbital angular momentum or spin angular momentum, but with total angular momentum. When the particle is at rest, however, $\mathbf{p} = 0$, and so

$$[S_i, H] = [\tilde{\alpha}_i/2, H] = -i \ {}_{ijk}\alpha$$

Taking the sum and the difference of these equations, we get

$$(p^0 - \sigma \cdot \mathrm{p})(u_1 + u_2) = m(u_1 - u_2),$$
$$(p^0 + \sigma \cdot \mathrm{p})(u_1 - u_2) = m(u_1 + u_2).$$

By defining

$$u_l = \frac{1}{2}(u_1 - u_2), \tag{8}$$
$$u_r = \frac{1}{2}(u_1 + u_2), \tag{9}$$

our equations can be rewritten as

$$(p^0 - \sigma \cdot \mathrm{p})u_r = mu_l,$$
$$(p^0 + \sigma \cdot \mathrm{p})u_l = mu_r.$$

These equations are coupled via the mass term. By letting mass go to zero, we have the uncoupled equations

$$p^0 u_r = \sigma \cdot \mathrm{p} u_r, \tag{10a}$$
$$p^0 u_l = -\sigma \cdot \mathrm{p} u_l. \tag{10b}$$

Since mass is a Lorentz scalar, these equations are Lorentz covariant. However,

2.3 On chirality

Hence any 2-component spinor can be uniquely decomposed into components of these operators, which project into spaces of positive and negative helicity. This notion can be generalized to 4-component spinors by defining the following helical projection operator:

$$P^{(\pm)}_{4\times 4} = \begin{pmatrix} P^{(\pm)} & 0 \\ 0 & P^{(\pm)} \end{pmatrix}. \tag{20}$$

It is trivial to check that

$$[P_{R,L}, P^{(\pm)}_{4\times 4}] = 0,$$

which implies that spinors can be simultaneous eigenstates of chirality and helicity in the massless limit.

We have seen that, in the massless limit, particles of positive chirality have positive helicity, and likewise, particles of negative chirality have negative helicity. But this is only true for massless particles. Chirality can be thought of as an inherent trait of particles, whereas helicity depends on the momentum of a particle. For massive particles, it is possible to Lorentz boost to different frames of reference to change helicity. But this is not true for massless particles; hence chirality corresponds exactly to helicity for massless particles.

2.4 Properties of eigenstates of the helicity projections

In this section, we will derive properties of right-handed spinors and state the analogous properties of left-handed spinors, since their derivation is nearly identical.

Write positive and negative energy solutions to the massless Dirac equation with right-handed chirality as

$$u^{(+)} = \frac{1}{2}(\mathrm{I} + \sigma \cdot \hat{\mathrm{p}})\tilde{u},$$
$$v^{(+)} = \frac{1}{2}(\mathrm{I} + \sigma \cdot \hat{\mathrm{p}})\tilde{v}.$$

Choose

$$\tilde{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then the positive and negative energy solutions can be written explicitly as

$$u^{(+)}(\mathrm{p}) = \frac{1}{\sqrt{2|\mathrm{p}|(|\mathrm{p}| + p_3)}} \begin{pmatrix} |\mathrm{p}| + p_3 \\ p_1 + ip_2 \end{pmatrix},$$
$$v^{(+)}(\mathrm{p}) = \frac{1}{\sqrt{2|\mathrm{p}|(|\mathrm{p}| - p_3)}} \begin{pmatrix} p_1 - ip_2 \\ |\mathrm{p}| - p_3 \end{pmatrix}.$$

It is simple to check that these spinors satisfy

$$u^{(+)\dagger}u^{(+)} = v^{(+)\dagger}v^{(+)} = 1,$$
$$u^{(+)\dagger}(\mathrm{p})v^{(+)}(-\mathrm{p}) = v^{(+)\dagger}(-\mathrm{p})u^{(+)}(\mathrm{p}) = 0,$$
$$u^{(+)}u^{(+)\dagger} = v^{(+)}v^{(+)\dagger} = \frac{1}{2}(\mathrm{I} + \sigma \cdot \hat{\mathrm{p}}).$$

The 4-component spinors defined by (18) further satisfy

$$u_R^\dagger u_R = v_R^\dagger v_R = |\mathbf{p}|, \tag{21a}$$

$$u_R^\dagger(\mathbf{p}) v_R(-\mathbf{p}) = v_R^\dagger(-\mathbf{p}) u_R(\mathbf{p}) = 0, \tag{21b}$$

$$u_R u_R^\dagger = v_R v_R^\dagger = \frac{|\mathbf{p}|}{2} \begin{pmatrix} P^{(+)} & P^{(+)} \\ P^{(+)} & P^{(+)} \end{pmatrix}. \tag{21c}$$

Note that, with $p_0 = |\mathbf{p}|$, we can write

$$\begin{aligned}
\frac{1}{4} \not{p} \gamma^0 (\mathrm{I} + \gamma_5) &= \frac{1}{4} \begin{pmatrix} p_0 & -\sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & -p_0 \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix} \begin{pmatrix} \mathrm{I} & \mathrm{I} \\ \mathrm{I} & \mathrm{I} \end{pmatrix} \\
&= \frac{|\mathbf{p}|}{4} \begin{pmatrix} \mathrm{I} & \sigma\cdot\hat{\mathbf{p}} \\ \sigma\cdot\hat{\mathbf{p}} & \mathrm{I} \end{pmatrix} \begin{pmatrix} \mathrm{I} & \mathrm{I} \\ \mathrm{I} & \mathrm{I} \end{pmatrix} \\
&= \frac{|\mathbf{p}|}{2} \begin{pmatrix} P^{(+)} & P^{(+)} \\ P^{(+)} & P^{(+)} \end{pmatrix}.
\end{aligned}$$

Hence (21c) can be rewritten as

$$u_R u_R^\dagger = v_R v_R^\dagger =$$

In the non-relativistic limit, $|\mathbf{p}| \ll m$, and so $u_S \ll u_L$ and $v_S \ll v_L$. Hence the subscript S refers to the small component and the subscript L refers to the large component.

The positive energy solutions satisfy

$$\begin{pmatrix} m\mathbb{I} & \sigma \cdot \mathbf{p} \\ \sigma \cdot \mathbf{p} & m\mathbb{I} \end{pmatrix} \begin{pmatrix} u_L \\ u_S \end{pmatrix} = E \begin{pmatrix} u_L \\ u_S \end{pmatrix}, \tag{26}$$

which can be expanded to yield

$$\sigma \cdot \mathbf{p} u_S = (E - m) u_L, \tag{27a}$$

$$\sigma \cdot \mathbf{p} u_L = (E + m) u_S. \tag{27b}$$

By substituting (25a) into (27a), we find that

$$(\sigma \cdot \mathbf{p}) \frac{\sigma \cdot \mathbf{p}}{E + m} u_L = (E - m) u_L$$

This equation results in two equations given by

$$\sigma \cdot (\mathrm{p} - e\mathrm{A})u_{\mathrm{S}} = (E - m)u_{\mathrm{L}}, \tag{31a}$$

$$\sigma \cdot (\mathrm{p} - e\mathrm{A})u_{\mathrm{L}} = (E + m)u_{\mathrm{S}}. \tag{31b}$$

From (31b), we have

$$u_{\mathrm{S}} = \frac{\sigma \cdot (\mathrm{p} - e\mathrm{A})}{E + m}u_{\mathrm{L}} \approx \frac{\sigma \cdot (\mathrm{p} - e\mathrm{A})}{2m}u_{\mathrm{L}}, \tag{32}$$

in the non-relativistic limit. Substituting (32) into (31a) yields

$$[\sigma \cdot (\mathrm{p} - e\mathrm{A})]\,\frac{\sigma \cdot (\mathrm{p} - e\mathrm{A})}{2m}u_{\mathrm{L}} \approx (E - m)u_{\mathrm{L}}. \tag{33}$$

Let us compute the products that appear in (33) to reduce notation. Note that

$$[\sigma \cdot (\mathrm{p} - e\mathrm{A})]$$

Notes

This paper represents a reproduction of my lecture notes on the properties of the Dirac equation, which are derived from Section 3 of Das' book on quantum field theory [1]. I have cut down much of the material and added some useful references to get a better picture of the utility and historical context of the content. Consider the text of this paper to be a script and the equations to be things that should be drawn on the chalkboard during the lecture.

References

[1] Ashok Das. *Lectures on quantum field theory*. World Scientific, 2008.

[2] Paul AM Dirac. The quantum theory of the electron. *Proc. R. Soc. Lond. A*, 117(778):610–624, 1928.

[3] Hermann Weyl. Gravitation and the electron. *Proceedings of the National Academy of Sciences*, 15(4):323–334, 1929.