1 Introduction

At this point in the course, the Klein-Gordon equation had been studied fairly thoro[redacted] the Lagrangian and Hamiltonian densities that would result in the Klein-Gordon[redacted] the field operator in terms of the creation and annihilation operators. However, w[redacted] the tools needed to derive this information, we made a key assumption in that we s[redacted] operator must be a real valued function. While this allowed us to develop many impo[redacted] it ignores many solutions to the equation. Therefore, it is my goal to show how t[redacted] purely real solutions could be applied when the field operator is complex.

2 Derivation of the Lagrangian and Hamiltonian [redacted]

2.1 Available Equations

As before, we have the original Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\varphi(x) = 0.$$

However, since we also know that $\varphi = \varphi^\dagger$, we also have

$$(\partial_\mu \partial^\mu + m^2)\varphi^\dagger(x) = 0.$$

We can also write φ and $\varphi^\dagger$ in terms of purely real functions by taking one real fu[redacted] part of φ and another to be the imaginary part. Therefore, we have

$$\varphi(x) = \frac{1}{\sqrt{2}}\left(\varphi_1(x) + i\varphi_2(x)\right)$$

$$\varphi^\dagger(x) = \frac{1}{\sqrt{2}}\left(\varphi_1(x) - i\varphi_2(x)\right).$$

We can also invert these relations in order to find φ_1 and φ_2 in terms of φ and $\varphi^\dagger$ [redacted]

$$\varphi_1(x) = \frac{1}{\sqrt{2}}(\varphi(x) + \varphi^\dagger(x))$$

$$\varphi_2(x) = \frac{-i}{\sqrt{2}}(\varphi(x) - \varphi^\dagger(x)).$$

By substituting (3) and (4) into (1) and (2), we can easily see that we have

$$(\partial_\mu \partial^\mu + m^2)\varphi_1(x) = 0$$

$$(\partial_\mu \partial^\mu + m^2)\varphi_2(x) = 0.$$

Therefore, a system that can be described by a compl(ac)-0.c62.85esc8(W)83.2(e)[redacted]

However, by factoring the terms in this Lagrangian density, we can rewrite it in terms of φ and $\varphi^\dagger$

$$\begin{aligned} L &= \frac{1}{2}\partial_\mu(\varphi_1 - i\varphi_2)\partial^\mu(\varphi_1 + i\varphi_2) - \frac{m^2}{2}(\varphi_1 - i\varphi_2)(\varphi_1 + i\varphi_2) \\ &= \partial_\mu\varphi^\dagger\partial^\mu\varphi - m^2\varphi^\dagger\varphi. \end{aligned} \tag{10}$$

We note that even though φ and $\varphi^\dagger$ are clearly not Hermitian, the Lagrangian density which describes this is in fact Hermitian.

2.3 Hamiltonian Density

2.3.1 φ_1 and φ_2

Now that we have the Lagrangian density, we can determine what the Hamiltonian density for this system

We can find the complex conjugate of the momentum operator in a similar fashion

$$\Pi^\dagger(x) = \frac{\partial L}{\partial \dot{\varphi}(x)} = \dot{\varphi}^\dagger$$
$$= \frac{1}{\sqrt{2}} (\Pi_1(x) - i\Pi_2(x)) \quad (16)$$

As before, we can determine the commutation relations of the system by noting that the φ and $\varphi^\dagger$ are independent variables.

$$[\varphi(x), \varphi(y)] = [\varphi(x), \varphi^\dagger(y)] = [\varphi^\dagger(x), \varphi^\dagger(y)] = 0 \quad (17)$$
$$[\Pi(x), \Pi(y)] = [\Pi(x), \Pi^\dagger(y)] = [\Pi^\dagger(x), \Pi^\dagger(y)] = 0 \quad (18)$$
$$[\varphi(x), \Pi^\dagger(y)] = [\varphi^\dagger(y), \Pi(y)] = i\delta^3(x - y) \quad (19)$$

The Hamiltonian density therefore can be written as

$$H = \Pi\dot{\varphi}^\dagger + \Pi^\dagger\dot{\varphi} - L$$
$$= \Pi^\dagger\Pi + \Pi^\dagger\Pi - \dot{\varphi}^\dagger\dot{\varphi} + \quad \varphi^\dagger \cdot \quad \varphi + m^2\varphi^\dagger\varphi$$
$$= \Pi^\dagger\Pi + \quad \varphi^\dagger \cdot \quad \varphi + m^2$$

$$J^{\mu} = \delta\varphi(x)\frac{\partial L}{\partial(\partial_{\mu}\varphi)} + \delta\varphi^{\dagger}(x)\frac{\partial L}{\partial(\partial_{\mu}\varphi^{\dagger})} - K^{\mu}, \tag{36}$$

where K^{μ} is defined by

$$L\,(\varphi\,(x), \partial_{\mu}\varphi\,(x)) - L\,(\varphi(x), \partial_{\mu}\varphi(x)) = \partial$$

Therefore, the $|k\rangle$ is an eigenvector of the charge operator with eigenvalue. A similar calculation can be done on $|\tilde{k}\rangle$ to determine that

$$Q|\tilde{k}\rangle$$

$$
\begin{aligned}
\langle 0|\varphi^\dagger(y)\varphi(x)|0\rangle &= \int\!\!\int \frac{d^3k\,d^3k'}{2(2\pi)^3\sqrt{k^0 k'^0}}\,\langle 0|\left[e^{-ik\cdot y}b(\mathbf{k}) + e^{ik\cdot y}a^\dagger(\mathbf{k})\right]\cdot\left[e^{-ik'\cdot x}a(\mathbf{k}') + e^{ik'\cdot x}b^\dagger(\mathbf{k}')\right]|0\rangle \\
&= \int\!\!\int \frac{d^3k\,d^3k'}{2(2\pi)^3\sqrt{k^0 k'^0}}\left[e^{-ik\cdot y}\langle 0|b(\mathbf{k}) + e^{ik\cdot y}\langle 0|a^\dagger(\mathbf{k})\right]\cdot\left[e^{-ik'\cdot x}a(\mathbf{k}')|0\rangle + e^{ik'\cdot x}b^\dagger(\mathbf{k}')|0\rangle\right] \\
&= \int\!\!\int \frac{d^3k\,d^3k'}{2(2\pi)^3\sqrt{}}
\end{aligned}
$$

$$V = m^2\varphi^\dagger\varphi + \frac{\lambda}{4}(\varphi^\dagger\varphi)^2 \tag{53}$$

By inspection, we can see that the minimum of the potential is 0, since it is composed of purely nonnegative terms. By considering the concept of a classical field (which we label as φ

$$\frac{\partial^2 V}{\partial \varphi_1^2} = -m^2 + \frac{3\lambda}{4}\left(\frac{4m^2}{\lambda}\right) = 2m^2$$
$$\frac{\partial^2 V}{\partial \varphi_2^2} = -m^2 + \frac{\lambda}{4}\left(\frac{4m^2}{\lambda}\right) = 0$$
$$\frac{\partial^2 V}{\partial \varphi_1 \partial \varphi_2} = 0 \qquad (60)$$

Since the second derivative is negative when $\varphi_{1c} = \varphi_{2c} = 0$ and positive when $\varphi_{1c} = \frac{\pm 2m}{\sqrt{\lambda}}$ and $\varphi_{2c} = 0$, the first point corresponds to a local maximum, and the second to a local minimum. This can be seen more clearly in the figure 1. This figure shows the potential for this system, often called the Mexican hat potential.

Figure 1: The Mexican hat potential which represents the complex Klein-Gordon equation [2]

7 Conclusion

We can now see that the complex solutions to the Klein-Gordon equation provide a wealth of information that cannot be found simply from the real solutions. The complex solutions allow for an interpretation of anti-particles that does not require negative energies, making it much more attractive than Dirac's hole theory. These solutions also allow the Feynman Green's function to defined in terms of the vacuum state. Lastly, the potential associated with the complex fields is one that leads to spontaneous symmetry breaking, the details of which are unfortunately outside the scope of this paper.

References

[1] Das A. 2008. *Lectures on Quantum Field Theory*, World Scientific, 2008, First Edition, pp. 257-75.

[2] Miller, Rupert `https://upload.wikimedia.org/wikipedia/commons/8/83/Mexican_hat_potential_polar_with_details.svg`