QUANTUM INFORMATION THEORY

Introducing the Bipartite Quantum System: the Density Matrix and the Bloch Sphere

Stuti Shah

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University of Rochester

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Abstract

...In quantum universe, there are no such things as accidents; only possibilities folded into existence by perception.

- J. Michael Straczynski

\Much is currently made of the concept of information in physics, following the rapid growth of the elds of quantum information theory and quantum computation. These are new and exciting elds of physics whose interests for those concerned with the foundations and conceptual status of quantum mechanics are manifold. On the experimental side, the focus on the ability to manipulate and control individual quantum systems, both for computational and cryptographic purposes has led not only to detailed realization of many of the gedanken experiments familiar from foundational discussions but also to wholly new demonstrations of the oddity of the quantum world. Developments on the theoretical side are no less important or interesting. Concentration of the possible ways of using distinctively quantum mechanical properties of systems for the purposes of carrying and processing information has led to considerable deepening of our understanding of quantum theory. The study of the phenomenon of entanglement, for example, has come on in leaps and bounds under the aegis of quantum information." [1] This paper is based on the lecture I gave for the Kapitza Society, where the objective this semester was to introduce ourselves to the emerging eld of quantum information theory.

1 Introduction

With the assumption that the reader is familiar with the fact that the state of an isolated quantum system can be represented by a vector in the state space, we shall see, in this paper, that it can also be represented (more conveniently) by a Hermitean operator known as the `density operator' [2] We shall descry that the density operator is a mathematical tool devised to not only represent the state of an isolated system, but also the state of a system that interacts signi cantly with its environment, or even the state of an ensemble of systems prepared in di erent ways.

In this paper, we aim to grasp this profound abstraction (to a modest degree), as it simpli es mathematical formalism and prove the basic properties of a density operator. We later look at a way to represent, derive, and deduce information about a quantum state in a 2-level system through the Bloch sphere[3].

2 The Density Matrix

Richard Feynman (physicist, 1918-1988) once said that while performing experiments or tackling quantum mechanical problems, we divide the universe into two parts: the rst, about which we have (some) information and we're interested in investigating more about, and the second, everything else The rst part, the part that we focus on, is our `system' and if it can be described by a single vector, then it is in what is known as a `pure state'. Now consider our system being connected to a bath and it is known to interact with it in some de nable manner such that the e ects of the interaction cannot be neglected while making measurements on the system. The `bath' here is a collection of random variables that uctuate in a way we have no information about - it is the part of the universe that is outside the system but still of signi cance to us. Being able to express all the information contained by a system (which is now in a `mixed state') in such a bath is our motivation to formalize and use density matrices.

32 22 22 Suppose that we're studying the phenomenon of polarization of light. And it is known that the

22 32 light we're using can be`mixedwanner`2(con)28llycanevWus.80(Let'scan)-3s28(a)28(t)-345(w)28(t)-333(otene

22 32 32 22 2 22 32 32

For $\vec{h}$ and $\vec{v}$ for instance:

$$\hat{\rho}_h = \vec{h}\vec{h}^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{2}$$

$$\hat{\rho}_v = \vec{v}\vec{v}^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{3}$$

Note that the respective vectors are enough to contain all the information about the "states" that the matrices above do and therefore it is verified that $\vec{h}$ and $\vec{v}$ are pure states.

The density operator formalism allows the treatment of pure states as special cases of statistical mixtures. We may say that if we know with certainty that the system is in the pure state $|\psi\rangle$, we can represent that state by a statistical mixture having $|\psi\rangle$ as its sole element with the assumption that the state has norm unity; its density operator is the projector operator (outer product).

For some quantum state, say a two-level spin system $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$ with $|a|^2 + |b|^2 = 1$ (normalization condition) and $(a; b$

hold this information and to describe đ, since đ is now in a mixed state.

Thus, those matrices that provide us with information such as the probability of nding the system in a particular state, and how the interaction between the system and the bath takes place

2) There is no global phase ambiguity.

Consider a state $|\psi\rangle$. The corresponding density operator $\rho = |\psi\rangle\langle\psi|$. Now consider another state $|\psi'\rangle = e^{i\theta}|\psi\rangle$. The corresponding density operator,

$$\rho' = |\psi'\rangle\langle$$

Proof for a pure state:

$$
\begin{aligned}
\mathrm{tr}\,(\hat{\ }) &= \sum_i {}_{ii} \\
&= \sum_i h i j \ \ i h \ \ j i i \\
&= \sum_i h \ \ j i i h i j \ \ i \\
&= h \ \ j \ \ i \\
&= 1
\end{aligned}
\tag{12}
$$

Proof for a mixed state:

$$
\begin{aligned}
\mathrm{tr}\,(\hat{\ }) &= \sum_i \sum_k p_k \, h i j \ \ i h \ \ j i i \\
&= \sum_k p_k \sum_i h \ \ j i i h i j \ \ i \\
&= \sum_k p_k \\
&= 1
\end{aligned}
\tag{13}
$$

Notice that the diagonal entries in equation 5 add up to 1 (as they should) $\frac{1}{2} + \frac{1}{2} = 1$. Also note that equation 4, this property gets us back to the normalization condition.

5) $\hat{\ }^2 = \hat{\ }$

Proof (for a pure state):

$$
\begin{aligned}
\hat{\ }^2 &= j \ \ i h \ \ j \ \ i h \ \ j \\
&= j \ \ i h \ \ j \\
&= \hat{\ }
\end{aligned}
\tag{14}
$$

Since we're dealing with statistical mixtures, an important result to note is the expectation value for an observable A in terms of the density operator, $hAi = \mathrm{tr}\,(\hat{A}\hat{\ })$.

Proof for a pure state:

$$
\begin{aligned}
\langle A \rangle &= \langle \psi | \hat{A} | \psi \rangle \\
&= \sum_{i,j} \langle \psi | i \rangle \langle i | \hat{A} | j \rangle \langle j | \psi \rangle \\
&= \sum_{i,j} A_{ij} \rho_{ji} \\
&= tr(\hat{A}\hat{\rho})
\end{aligned}
\tag{15}
$$

For a mixed state,

$$
\begin{aligned}
\langle A \rangle &= \sum_{k} p_k \langle \psi_k | \hat{A} | \psi_k \rangle \\
&= \sum_{i,j,k} p_k \langle \psi_k | i \rangle \langle i | \hat{A} | j \rangle \langle j | \psi_k \rangle \\
&= \sum_{i,j} \langle i | \hat{A} | j \rangle \sum_{k} p_k \langle j | \psi_k \rangle \langle \psi_k | i \rangle \\
&= \sum_{i,j} A_{ij} \rho_{ji} \\
&= tr(\hat{A}\hat{\rho})
\end{aligned}
\tag{16}
$$

2.2 The Bloch Sphere

\How does real three dimensional space that we live in, correspond to the two dimensional complex vector space within which a qubit sits? Answering this question is really important if we want to manipulate the state of a qubit in a three dimensional real space. It turns out, mathematically at least, there's a very beautiful answer to this question which comes from the Bloch sphere representation of a qubit." (Sandro Mareco)

The Bloch sphere is a geometrical representation of a quantum state in a 2-level system. It gives

+1) of σ_x) and $\sigma_x|\downarrow_x\rangle = -1|\downarrow_x\rangle$ ($|\downarrow_x\rangle$ is an eigenvector (with eigenvalue -1) of σ_x); $|\uparrow_x\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $|\downarrow_x\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$. Thus, the answer is to find that state which is an eigenvector of $\hat{n}\cdot\vec{\sigma}$ with eigenvalue +1. And we already know what that state is ($|\uparrow_n\rangle$)! Let us check!

$$
\begin{aligned}
(\hat{n}\cdot\vec{\sigma})(|\uparrow_n\rangle) &= \begin{pmatrix} \cos(\theta) & e^{-i\phi}\sin(\theta) \\ e^{i\phi}\sin(\theta) & -\cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\frac{\theta}{2}) \\ e^{i\phi}\sin(\frac{\theta}{2}) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta)\cos(\frac{\theta}{2}) + e^{-i\phi}\sin(\theta)e^{i\phi}\sin(\frac{\theta}{2}) \\ e^{i\phi}\sin(\theta)\cos(\frac{\theta}{2}) - \cos(\theta)e^{i\phi}\sin(\frac{\theta}{2}) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta - \frac{\theta}{2}) \\ e^{i\phi}\sin(\theta - \frac{\theta}{2}) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\frac{\theta}{2}) \\ e^{i\phi}\sin(\frac{\theta}{2}) \end{pmatrix} \\
&= +1\,|\uparrow_n\rangle
\end{aligned}
\tag{21}
$$

The same follows for $|\downarrow_n\rangle = \begin{pmatrix} \sin(\frac{\theta}{2}) \\ -e^{i\phi}\cos(\frac{\theta}{2}) \end{pmatrix}$ i.e. $(\hat{n}\cdot\vec{\sigma})(|\downarrow_n\rangle) = -1|\downarrow_n\rangle$.

Note that $\langle\uparrow_n|\uparrow_n\rangle = 1$, $\langle\downarrow_n|\downarrow_n\rangle = 1$, and $\langle\uparrow_n|\downarrow_n\rangle = 0$.

Some sanity checks using for example $\hat{n} \doteq (1;0;0)$, $\theta = \frac{\pi}{2}$, and $\phi = 0$ can be used to verify the above results. For $\theta = \pi - \theta$ and $\phi = \phi + \pi$ we find that $|\uparrow_{-n}\rangle = |\downarrow_n\rangle$.

The expression for the Bloch vector can similarly be verified for:

$\hat{n}_x$

$$\langle \uparrow_z | \uparrow_n \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\frac{}{2}) \\ e^{i} \sin(\frac{}{2}) \end{pmatrix} = \cos(\frac{}{2})$$

$\Rightarrow$ the probability of spin-up in z direction is $|\langle \uparrow_z | \uparrow_n \rangle|^2 = \cos^2(\frac{}{2}) = \frac{1}{2}(1 + \cos( ))$.

Thus, for any two unit vectors $\hat{n}$ and $\hat{m}$ with an angle between them, we have

$$P = |\langle \uparrow_m | \uparrow_n \rangle|^2 = \frac{1}{2}(1 + \hat{n} \;\; \hat{m}) \tag{22}$$

The probability is 1 for $\hat{n} \;\; \hat{m} = 1$, $\frac{1}{2}$ for $\hat{n} \;\; \hat{m} = 0$, and 0 for $\hat{n} \;\; \hat{m} = \;\; 1$.

Thus the state of a qubit can be represented in terms of two parameters and as follows:

$$| \;\; \rangle = \cos \frac{}{2} \, |0\rangle + e^{i} \sin \frac{}{2} \, |1\rangle \tag{23}$$

where 0 and 0 2 .

We know that the state of a qubit should be $| \;\; \rangle = \;\; |0\rangle + \;\; |1\rangle$, where $(\; ; \;) \in \mathbb{C}$ and $| \;\; |^2 + | \;\; |^2 = 1$.

Representing a complex number in its polar form, we have that $= re^{i}$. Thus,

$$| \;\; \rangle = r_0 e^{i \;\; 0} |0\rangle + r_1 e^{i \;\; 1} |1\rangle \tag{24}$$

where $(r_0; r_1)$ are non-negative real numbers.

$| \;\; \rangle = e^{i \;\; 0}$[Td [(i)]TJ/F28 10.9091 Tf 153.608 0 Td [((Tf 5.455 0 Td 9.157 0 T8

3 Summary

In this paper we introduced ourselves to the density operator, we looked at the various properties of a density operator, and saw the representation of a qubit on the Bloch sphere.

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