SHIFTED K-THEORETIC POIRIER-REUTENAUER ALGEBRA

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1. Introduction

Poirier and Reutenauer defined a Hopf algebra on the Z-span of all standard Young tableaux in [10], which is later studied in [4, 11]. The Robinson-Schensted-Knuth insertion was used to relate the bialgebra to Schur functions. Schur function is a class of symmetric functions that can be determined by the summation of all semistandard Young tableaux of shape λ. With the help of the PR-bialgebra, the Littlewood-Richardson rule is established, which gives an explicit description on the multiplication of arbitrary Schur functions. The generalization of this approach has been used to develop the Littlewood-Richardson rule for other classes of symmetric functions. In [9], a K-theoretic analogue is developed using Hecke insertion, providing a rule for multiplication of the stable Grothendieck polynomials. Similarly, in [6], a shifted analogue is developed, providing a rule for multiplication of P-Schur functions. We use a shifted Hecke insertion, introduced in [8], to develop a shifted K-theoretic and v are (possibly empty) words of positive integers, and $a < b < c$ are distinctive positive

First, proceed with how to insert a positive integer x into a given shifted increasing tableau T. Start with inserting x into the first row of T. For each insertion, assign a box to record where the insertion terminates. This data will be used when the recording tableau is introduced in Subsection 2.4.

The rules for inserting x to T are as follows:

(1) If x is weakly larger than all integers in the row (resp. column) and adjoining x to the end of the row (resp. column) results in an increasing tableau T , then T is the resulting tableau. We say the insertion terminates at the new box.

Example 2.6. Inserting 4 into the first row of the left tableau gives us the right tableau below. The insertion terminates at box $(1, 4)$.

1	2	3
	3	5
		6

1	2	3	4
	3	5	
		6	

(2) If x is weakly larger than all integers in the row (resp. column) and adjoining x to the end of the row (resp. column) does not result in an increasing tableau, then $T = T$. If x is row inserted into a nonempty row, we say the insertion terminated at the box at the bottom of the column containing the rightmost box of this row. If x is row inserted into an empty row, we say that the insertion terminated at the rightmost box of the previous row. If x is column inserted, we say the insertion terminated at the rightmost box of the row containing the bottom box of the column x could not be added to.

Example 2.7. Adjoining 4 to the first row of the left tableau does not result in an increasing tableau. Thus the insertion of 4 into the first row of the tableau on the left terminates at (2,3) and gives us the tableau on the right.

1	3

Definition 2.12. [8, Definition 5.16] A shifted K-theoretic jeu de taquin is only used to show that the shifted Hecke insertion is consistent with the weak K-Knuth equivalence on words, we omit the proof here.

Theorem 2.16. *[5, Corollary 2.18] If* $P_{SK}(u) = P_{SK}(v)$, *then* $u \hat{\equiv} v$.

From this point on, we refer to both the weak K-Knuth equivalence on words and the weak K-Knuth equivalence on insertion tableaux as ”equivalence”

Remark 2.17.

(1) [2, Corollary 7.2] *Minimal increasing shifted tableaux are URTs.*
(2) [3, Theorem 1.1]

Now we define the product of $[[h]]$ and $[[h']]$ in $SKPR$

$$[[h]] \cdot [[h']] = \sum_{w \triangleq h, w' \triangleq h'} w \sqcup\!\sqcup w' [n].$$

where h is a word in the alphabet $[n]$ and h' is a word in the alphabet $[m]$.

The following theorem shows that this product is a binary operation on the vector space $SKPR$.

Theorem 3.2. *The product of any two initial words h and h'*

For each of these tableaux, restricting to the alphabet $\{1, 2\}$ gives the tableau $P_{SK}(12)$. Also, the reading word of each restricted to the alphabet $\{3, 4\}$ is weak K-Knuth equivalent to $h[2] = 34$.

Corollary 3.6. *If both T_1 and T_2 are URTs, then*

$$\left(\sum_{P_{SK}(u)=T_1} u\right) \cdot \left(\sum_{P_{SK}(v)=T_2} v\right) = \sum_{T \quad T(T_1 \sqcup}$$

Example 4.4. We have

$$K_{(2,1)} = x_1^2 x_2 + 2x_1 x_2 x_3 + 3x_1^2 x_2^2 + 5x_1^2 x_2 x_3$$

Finally, let $(j$

Proof.

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