Crushtaceans and Complements of Fully Augmented and Nested Links

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Abstract

This paper addresses fully augmented links (FALs) and nested links, two subclasses of generalized FALs. We utilize a graph called the crushtacean,

A second question that Magnum and Stanford ask is \Can we characterize those links that are determined by their complements?" In Section 3.2, we prove that

Figure 3: The cell decomposition process for a nested link complement.

where every vertex has degree three. A graph is *maximal planar* if it is simple and planar, but adding any edge would destroy one of these properties.

A *triangulation of* S^2 is a simple planar graph with at least two faces such that each face has three edges and no distinct faces share more than one edge. Purcell proves in [10] that the crushtacean is the dual graph to a triangulation of S^2. We now investigate combinatorial properties of spherical triangulations, which combine with duality to reveal similar structure in crushtaceans.

Lemma 1.3.1. *Let be a triangulation of* S^2*. Then, has at least 4 vertices and is maximal planar.*

Proof. Let be a triangulation of S^2. Since is simple, if had fewer than three vertices, then must have fewer than two faces. Thus, must have at least three vertices. Now, suppose has exactly three vertices. Then, since it is simple, planar, and must have more than one face, must form a triangle, splitting S^2 into exactly two faces. These two faces must share more than one edge. This is a contradiction, so must have more than three vertices. This proves the rst half of our statement.

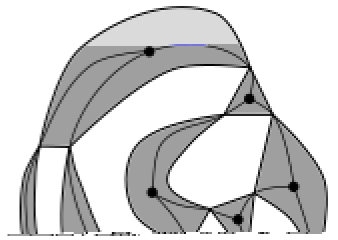


Figure 4: Constructing a crushtacean.

To add any edge to while maintaining planarity, we must add an edge through a face. Since each face of is a triangle, however, if vertices u and v share a face, then u and v must also be adjacent. Thus, we cannot add any edges to while maintaining both planarity and that is simple, so must be maximal planar. □

A graph G is *k-vertex-connected* if G has more than k vertices and remains connected when any collection of fewer than k vertices are removed. It is a common fact, stated in [3], that maximal planar graphs with at least four vertices are 3-vertex-connected. We have then shown that any triangulation of S^2 is 3-vertex connected. A 3-vertex connected, simple, planar graph is also known as a *polyhedral graph*.

Proposition 1.3.2. *The crushtacean of a hyperbolic nested link is 3-vertex-connected, simple, planar, trivalent graph.*

Proof. Let be the crushtacean of a hyperbolic nested link. Since the upper half of each twice punctured disk in the cell decomposition has three edges, we know each shaded region of the decomposition will be a triangle. Then, since we connect vertices of through the corners of these shaded regions, each vertex must be degree three.

In [10], Purcell proves that the crushtacean must be the dual graph of a triangulation of S^2, which we have shown must be a polyhedral graph. The dual to

A *balanced tree* is a tree that admits an involution that xes one edge, e. This involution induces a coloring such that edges mapped to one another have the same color. We'll refer to the edge e as the *edge of symmetry*. Now, given a graph G, a *balanced spanning forest* on G is a collection of disjoint balanced trees in G such that every vertex of G lies in some tree in this forest.

From the cell decomposition of the complement of a nested link, we can nd a unique balanced spanning forest on the crushtacean. This will be constructed in such a way so that vertices mapped to each other by the involution correspond to that a pair of vertices in a crushtacean are *glued* if they are mapped to one another by the involution on a balanced tree, in connection to the corresponding faces in the cell decomposition being glued. The *gluing pattern* refers to the collection of information about which vertices are glued. Finally, a pair of vertices will be said to be *glued by* a tree or forest if they are glued in the induced gluing pattern.

2 Nested Links with the Complement of an FAL

This section focuses on describing a su cient condition for the complement of a nested link to be homeomorphic to that of a fully augmented link. Subsection 2.1 provides a brief introduction to Dehn twists in the context of link complements. Subsection 2.2 describes conditions for the gluing pattern on a nested link painted crushtacean to be the same as that for an FAL painted crushtacean. In cases where the gluing pattern is the same, a homeomorphism between the complements of these nested links and FALs is then described. Subsection 2.3 introduces a family of crushtaceans called *generalized ladder graphs* that have the properties described in Subsection 2.2. A complete description of this family is given.

2.1 Dehn Twists

in Whitehead's construction is to slice along such a disk D, rotate one full time, and then re-glue the disk, as is shown in Figure 7. This process of slicing along a disk bounded by an unknotted component, rotating one full time, and regluing is called an *Dehn twist*. Intuitively, one can think of this process as adding one full twist between the strands that pass through C and puncture D. As they have been described, a Dehn twist is a homeomorphism between link complements, as is discussed in [11].

Figure 7: The steps of a Dehn twist.

2.2 Ladder Subgraphs of Crushtaceans

Given a link L, its complement S^3nL is a 3-manifold. If a link L is isotopic to L, then S^3nL is homeomorphic to S^3nL , but the converse is not necessarily true. Our goal is to address when nested links and fully augmented links have homeomorphic complements. The following Theorem concerning simple planar trivalent graphs will be useful in considering the question.

Theorem 2.2.1. *Given a simple trivalent graph with a balanced spanning forest, there exists a perfect matching that induces the same gluing pattern if and only if all vertices are glued to an adjacent vertex in the gluing pattern induced by the forest.*

Proof. If the vertices glued by the balanced spanning forest are not adjacent, then no perfect matching can induce the same gluing pattern, as all vertices glued by a perfect matching are adjacent.

Now, to address the converse, we suppose that all pairs of glued vertices are adjacent. We'll show that the collection of these edges of adjacency between glued vertices give a perfect matching on the graph. First, notice that none of these edges of adjacency can share a vertex, as every vertex is glued to exactly one other. Second, since every vertex is glued to another, this collection of disjoint edges span the vertices of the graph, hence is a perfect matching. □

Now, we want to introduce a certain type of graph called a *ladder graph*, denoted L_n, with $2n$ vertices and $3n$ 2 edges. We depict L_5 and the general structure of L_n

The edge of symmetry that is also in the perfect matching is colored red in Figure 9. In this diagram, we have colored one adjacent edge blue to represent the continuing tree. This tree must be balanced, so we must also color either e_1 or e_2 blue. We'll look at these as two separate cases.

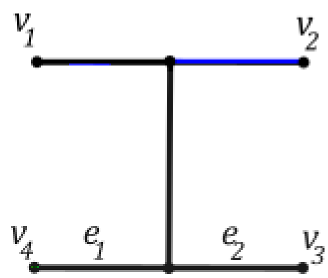


Figure 9: An edge of symmetry with one adjacent edge colored.

Case 1: Assume we color e_1 blue. This tree must then glue the vertices v_2 and v_4, but then our hypothesis and Theorem 2.2.1 tell us that there must be an edge between v_2 and v_4, as depicted in the first diagram of Figure 10. Now, we have two subcases: either $v_1 \neq v_4$ or $v_1 = v_4$ (where equality here means that they are the same point).

If $v_1 \neq v_4$, then there must be some subgraph T that v_1 is connected to, as shown in the second image of Figure 10. Then since our graph must be 3-vertex connected, by Proposition 1.3.2, there must be an edge connecting T to each of v_2 and v_4, else there would be two vertices that we could remove and disconnect T from the rest of Figure some subgraph G that would be disconnected from the crushtacean by removing v_2 and v_5, as in the second diagram of Figure 11, which would again contradict that the crushtacean is 3-vertex-connected. Thus, in this case, our crushtacean must be K_4.

Figure 11: Addressing the case where e_1 is colored and $v_1 = v_4$.

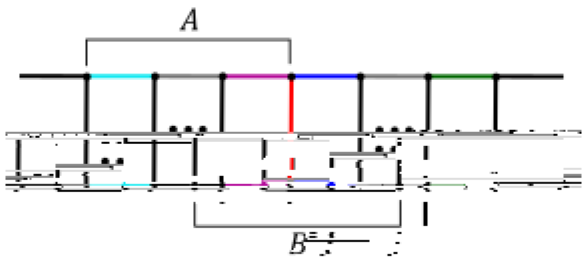


Figure 13: A large ladder.

some examples for a ladder with two rungs. Note that in general, a ladder with n rungs with the canonical perfect matching will correspond to a chain with n crossing circles. Figure 15 depicts this, along with the same ladder and an arbitrary balanced tree that gives the same gluing pattern.

Figure 14: Top: A two rung ladder with the canonical perfect matching and the associated tangle. Bottom: A two rung ladder with a single balanced tree that gives the same gluing pattern as above and the associated tangle.

Theorem 2.2.3. *If a fully augmented link F and a nested link N have the same crushtacean, other than K_4, with the same gluing pattern, then there is a homeomorphism $h : S^3 \setminus F \to S^3 \setminus N$ given by a sequence of Dehn twists.*

Proof. By Proposition 2.2.2, the balanced forest associated with N must only differ from the perfect matching associated with F

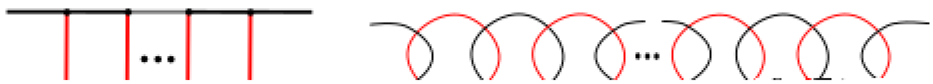


Figure 15: Top: An n rung ladder with the canonical perfect matching and the associated tangle. Bottom: An n rung ladder with a single balanced tree that gives the same gluing pattern as above and the associated tangle.

with F on a single ladder. The more general case will be a composition of such homeomorphisms.

First, notice that we can also describe h as a composition of homeomorphisms h_i between the complements of nested links corresponding to iteratively removing one perfect matching edge from the forest and extending the balanced tree on this ladder by two rail edges that will then glue the same vertices as the removed edge.

Now, note that each homeomorphism h_i will be isotopic to the identity everywhere except the region local to the tangle associated with this ladder, since the links F and N must be identical on all other regions. We'll now proceed inductively, to show that each h_i is given by a sequence of Dehn twists.

The base case will be replacing two rungs of a ladder with a balanced tree that gives the same gluing pattern, as is depicted in Figure 14. By symmetry, examining this case will also su ce to address the case where the edge of symmetry is on the right of the colored rails. We will now use Figure 16 to depict the sequence of Dehn twists that will give the homeomorphism h_1. The rst image in this gure gives a sublink of F as follows: for our current purposes, the green torus represents the

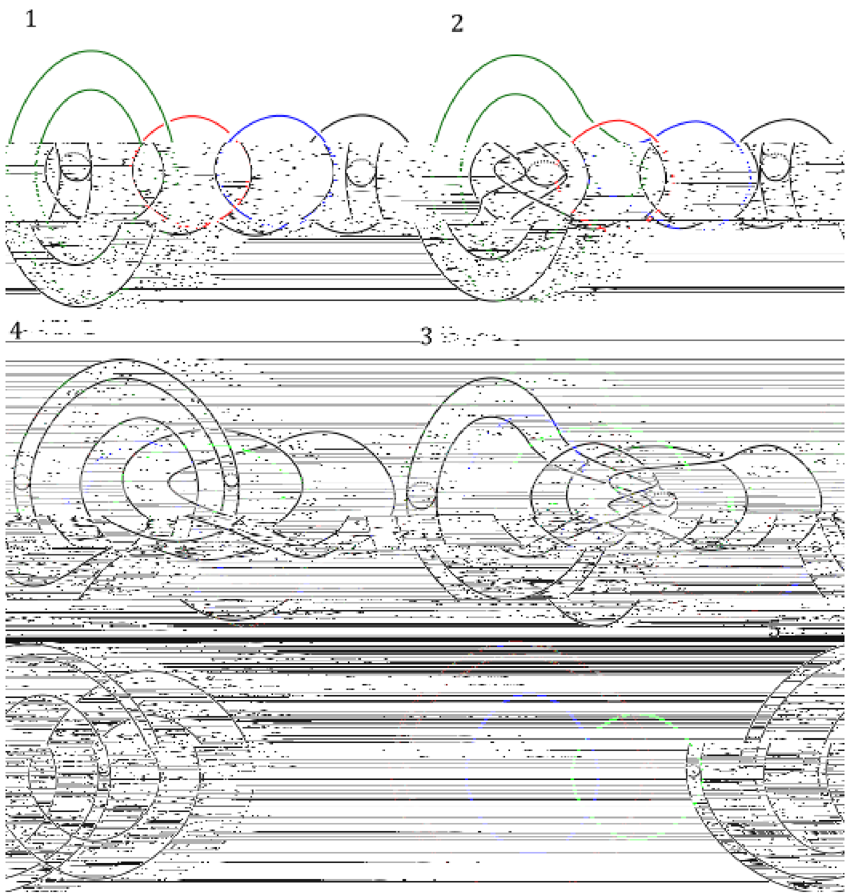
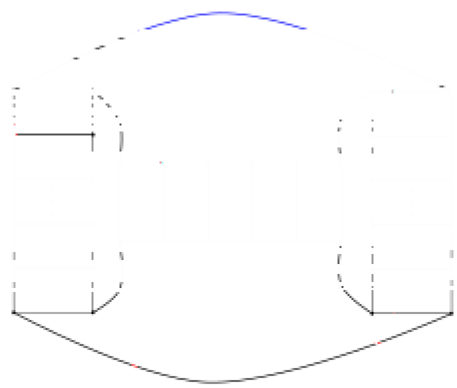


Figure 16: The Dehn twists described in the proof of Theorem 2.2.3.

and we perform an isotopy in the fourth image to emphasize that these components are indeed unlinked. Finally, in the fifth image, we do a single Dehn twist about the red component, which unlinks the black and the blue components. One can then confirm with Figure 14 that the result of these Dehn twists matches the expected tangle.

Now, assume h_i is given by a sequence of Dehn twists for $1 \le i \le n$. We'll show that h_{n+1} is also given by Dehn twists. We'll again assume that the tree is being extended on the right side, and the other case will be given by symmetry. We will again reference Figure 16 (however the green torus will be interpreted differently in this step). The sublink we'll look at this time is given in Figure 17. It will suffice to look at this sublink because we have included all components that link with circles u and v, which will be the only components that we perform Dehn twists about. In this case, we'll take the meridian of the green torus to be the knot circle labeled *c* that lies in the plane. The torus will then encompass the crossing circles that appear green, blue, and red in Figure 17, in addition to those that lie in the ellipsis. The red and blue components in Figure 16 will correspond to those labeled *u* and *v* in Figure 17, respectively.

We will again start by performing one Dehn twist about the red component. Now, however, the components contained in the green torus will become twisted be called *feet* (with the singular *foot*). An example of such a structure is depicted in Figure 18, where the feet are colored blue.



removed disconnects the graph. Similarly, we'll de ne a *bridge edge* of an arbitrary graph as an edge that if removed disconnects the graph.

Lemma 2.3.1. *A generalized ladder graphs with at least two ladders is 3-vertex connected if and only if no foot edge connects vertices that lie in the same ladder and there is no bridge ladder.*

Proof. First, suppose there is a foot of a generalized ladder graph that connects two vertices of the same ladder. We require that generalized ladder graphs be simple, so this foot must not connect two verties that are already adjacent in the ladder itself. This must then happen as depicted in Figure 20. We notice, however, that since there is more than one ladder in our graph, the region labeled T is nontrivial. Thus, if we remove the two vertices circled in red, then our graph will become disconnected, so our graph is not 3-vertex-connected. Similarly, if there is a bridge

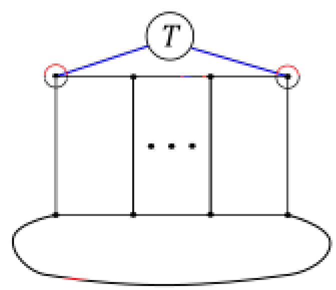


Figure 20: Subgraph of a generalized ladder graph where a foot connects two vertices in the same ladder.

ladder, then removing a pair of vertices that are endpoints of the same rung will clearly disconnect the graph.

Now, to address the converse, suppose every foot connects vertices in two distinct ladders and there is no bridge ladder. We'll show that in this case no pair of two vertices may be removed that disconnects our graph.

Proposition 2.3.2. *There is a bijection between the set of 3-vertex-connected generalized ladder graphs with more than one ladder and the set of perfect matchings on simple planar trivalent graphs without a bridge edge.*

Proof. Consider one colored edge, e, in a perfect matching on a simple planar trivalent graph without a bridge edge, G. Let v and v' be the endpoints of e. Let v_1 and v_2 be the other vertices adjacent to v and let v_3 and v_4 be the other vertices adjacent to v'. Now, we remove $e; v$, and v' from our graph. Then, we add in a ladder with an arbitrary number of rungs such that there is a foot connecting each of v_1, v_2, v_3, and v_4 to one of the degree 2 vertices in our ladder so that v_1 and v_2 are adjacent and v_3 and v_4 are adjacent. This should be done so that these new

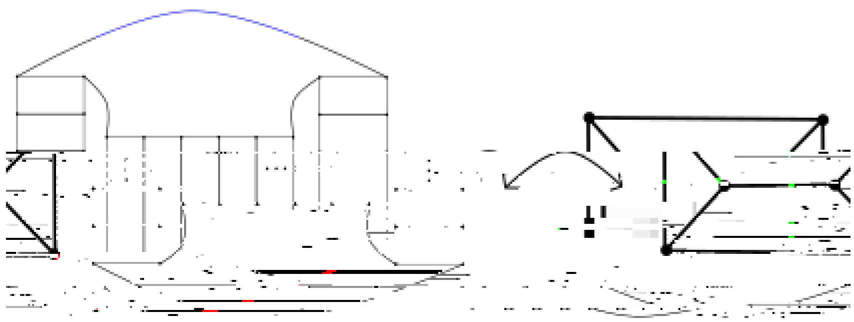


Figure 21: The correspondence between a generalized ladder graph and a simple planar trivalent graph with a perfect matching (in red).

simple planar trivalent graphs is distinct from how we have used them before. This application describes a method to characterize generalized ladder graphs, while the former application describes the balanced spanning forest associated with an FAL.

3 Prism Graphs and Pretzel Links

This section focuses on applications to a family of FALs called fully augmented pretzel links. Subsection 3.1 describes a family of nested links with complements homeomorphic to the fully augmented pretzel links. The size of this family is shown to grow exponentially with the number of link components. In Subsection 3.2, we show that the fully augmented pretzel links are determined by their complements, within the class of fully augmented links.

3.1 Nested Links with the Complement of $S^3 n \mathcal{P}_n$

In this section, we'll look at the prism graphs, P_n with $2n$ vertices. These are of interest because they are the only 3-vertex-connected generalized ladder graphs with one ladder. The fully augmented link associated with P_n and the canonical perfect matching is what is sometimes referred to as the fully augmented pretzel link with n crossing circles, which we'll denote $\mathcal{P}_n$. Some properties of the fully augmented pretzel links are explored by Meyer, Millichap and Trapp in [8]. The prism graph P_n and the fully augmented pretzel link $\mathcal{P}_n$ are shown in Figure 22.

We would like to consider nested links with the same complement as $\mathcal{P}_n$. We will need to be able to determine when two balanced spanning forests on P_n that give the same gluing pattern as the canonical perfect matching are associated with two di erent links. We'll introduce some tools that will help us do this.

If c_1 and c_2 denote two components in a given link, let $lk(c_1; c_2)$ denote the

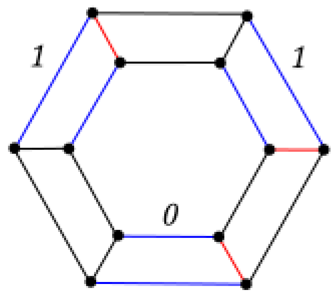


Figure 23: Example P_6 Coloring

Lemma 3.1.1. *Consider the link formed from a balanced spanning forest on P_n that gives the canonical gluing pattern. Suppose that on some connected sub-ladder two consecutive trees in the forest are composed of three edges. Then, the primary crossing circles associated with these trees each have a component linking number of three. Further, if in clockwise order we have*

i. a 0 tree followed by a 0 tree or a 1 tree followed by a 1 tree, then the associated primary knot

Figure 24: The four options for two consecutive trees with three edges.

Theorem 3.1.2. *For even n, there are at least* $2^{n=2}=n$ *distinct nested links whose complement is homeomorphic to that of* P_n*.*

Proof. Let P_n have a balanced spanning forest such that each tree has three edges and the gluing pattern is the same as that given by the canonical perfect matching in P_n. Then, it follows from Theorem 2.2.3 that the complement of the nested link associated with P_n has the same complement as P_n.

By Lemma 3.1.1, the primary crossing circles must have a component linking number of 3 and link with exactly two primary knot circles. Further, if any other components have a component linking number of 3, each must be a primary knot circle. Note that primary knot circles link with exactly two distinct primary crossing circles and that the sublink composed of primary crossing circles and primary knot circles forms a chain. Since component linking number is equal to the degree of the associated vertex in the linking graph, we have shown that there is a unique cycle in the linking graph where (at least) every other vertex has degree 3, given by vertices associated with primary crossing circles and primary knot circles, alternating. We'll look at the degree sequence along this cycle, which has length n.

If every vertex along this cycle has degree 3, then the crushtacean must be covered by all 0 trees or all 1 trees. Now, assume that not every vertex in this cycle has degree 3. Then, the vertices associated with primary crossing circles can be determined, since these all have degree 3 and alternate with the primary knot circles. Let's consider the subsequence given by the degrees of the primary knot circles. This subsequence will have length $n=2$, since every other vertex along the originally identi ed cycle is associated to a primary knot circle. We can traverse this subsequence in two directions. In each direction, we can associate an $n=2$ length cycle of binary digits, since the degree sequence of these vertices tells us when we change between 0 trees and 1 trees. Further, every binary cycle has an associated degree sequence of this form. There are $2^{n=2}$ binary strings of length $n=2$ and at most $n=2$ can correspond to the same cycle, so there are at least $2^{n=2}=(n=2)$ distinct binary cycles. In general, traversing the degree sequence in di erent directions will yield distinct binary cycles, so since there were two ways to traverse our degree sequence, we can say that there are at least $2^{n=2}=n$ binary cycles associated with distinct degree sequences. □

Note that this lower bound is not sharp. This argument can easily be extended by considering forests containing trees of depth other than two. Arguments can be made related to integer partitions of n. In this way, we can also consider odd n. The above argument is made to show that this value is at least exponential in $n=2$.

Theorem 2.2.3 tells us that all of these nested links are related by some sequence of Dehn twists. By the work of Whitehead, we know that, in general, we can use Dehn twists to generate in nitely many distinct links with homeomorphic complements [14]. Note, however, that we cannot continue to perform Dehn twists inde nitely and still be left with a nested link; this follows from the fact that there

n

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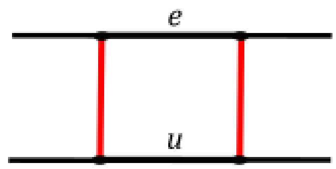

Figure 25: Two edges corresponding to one knot circle.

are only nitely many nested links with the same number of components.

3.2 Fully Augmented Pretzel Links

We end this section by taking a closer look at the fully augmented pretzel links.

Lemma 3.2.1. *The fully augmented pretzel links P_n (n 3) are the only at hyperbolic FALs with the same number of crossing circles and knot circles.*

Proof. First, note that there are no hyperbolic FALs with only one crossing disk. Now, consider a hyperbolic FAL with two crossing disks. The crushtacean for this link must have four vertices and must thus be K_4. Up to a rotation or re ection, there is only one perfect matching on K_4. This painting gives the Borromean rings

lwo ciossing sad thi fameumber cnot circle..

Note that this proof also tells us that P_n with the canonical perfect matching is the unique crushtacean for P_n.

Before proceeding to the statement and proof of the next theorem, we want to recall that Mostow's rigidity theorem tells us that geometric properties of nite-volume hyperbolic 3-manifolds are in fact topological invariants. In particular, the number of cusps and the volume of a hyperbolic link complement are topological invariants.

Theorem 3.2.2. *Within the class of hyperbolic at FALs, the fully augmented pretzel links P_n are uniquely determined by their complements.*

Proof. First, note that for each component in a hyperbolic link, there is an associated cusp in the complement. The number of cusps of a hyperbolic 3-manifold is a topological invariant, so if two hyperbolic links have homeomorphic complements, they must have the same number of components.

Suppose we have a hyperbolic at FAL F whose complement is homeomorphic to that of P_n. Note that P_n has $2n$ components, so we have shown F must have $2n$ components. We know that the number of crossing circles for any fully augmented link must be greater than or equal to the number of knot circles, so if F is distinct from P_n, then F must have at least $n + 1$ crossing circles, by Lemma 3.2.1. Our goal is to show that this cannot be the case, given that $S^3 \setminus F$ is homeomorphic to $S^3 \setminus P_n$.

In [10], Purcell states that if F has c crossing circles, then the hyperbolic volume of F is at least $2v_8(c - 1)$ where $v_8 = 3.66386\ldots$ is the volume of a regular ideal tetrahedron. We note that this lower bound is increasing with the number of crossing circles, so the lower bound for more than n

An earlier version of this paper utilized a \fact" that if two balanced spanning forests on a crushtacean gave the same gluing pattern, then the nested links associated with these balanced spanning forests must have homeomorphic complements. The reasoning follows from the correspondence between glued vertices and glued faces in the cell decomposition. Theorem 2.2.3 proves this for some cases, but a rigorous proof of the more general statement should be given elsewhere.

Finally, it has been conjectured that all FALs are uniquely determined by their complements within the class of fully augmented links. Theorem 3.2.2 proves this for the subclass of fully augmented pretzel links, but it does not seem likely that this particular method will generalize.

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