Equilateral triangles in vector spaces over nite elds

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Abstract

We show a subset of $\mathbb{F}_p^2$ of size $Cn^{\log_7(12)}$ does not necessarily contain any equilateral triangles by giving an explicit construction of such an equilateral-free subset. We do so by providing a map between $\mathbb{F}_p^2$ and sets of points on the plane. Lastly, we examine a special case of looking at only axis-aligned triangles.

Contents

1 Introduction

Various upper bounds have been proved for the maximum size of subsets $\mathbb{F}_p^d$ that do not contain certain con gurations of points. In (2) they show a upper bound on the maximum size of a subset of $\mathbb{F}_p^d$ which does not contain every at least $24p^{\frac{3+\ d}{2}}$ contains some equilateral triangle. Again, this gives a nontrivial bound only when $d > 3$.

A natural question is whether there exists any bound of the form Cp with some $< d$ when d is 2 or 3, or if it is possible to nd equilateral free sets with a positive percentage of the available points, or neither. For some values of p

distinct points. Three distinct points, $x; y; z$ 2 F_p^d are called an equilateral triangle i

$$jjx \quad yjj = jjx \quad zjj = jjy \quad zjj$$

and a subsetS F_p^d is called equilateral free if it has no such triple.

We will also be making use of the hexagonal coordinate system, which consists of all points of the form

$$f(a + \frac{b}{2}; \frac{b^{p}\bar{3}}{2})ja; b2 \ Zg$$

We can divide this into n^2 equivalency classes which we will calP_n with two points $(a_1 + \frac{b_1}{2}; \frac{b_1^{p}\bar{3}}{2})$ and $(a_2 + \frac{b_2}{2}; \frac{b_2^{p}\bar{3}}{2})$ in the same equivalency class i a_1 v. Td [(v. Td1.9626 Tf 107.009 0 equilateral triangles, and has been programmatically verified as the largest such subset of P_7. The code for this is in the Appendix. The code isn't fast enough to run for larger grids, but this proof method can use a better starting grid to get a better bound, if one is found.

Next, we will show that for any k and S, if there exists an equilateral-free subset of P_k with size S, how to construct one in P_{7k} with size $12S$. The method will be to place a P_7 into each point of the P_k, and in the S originally selected points, we select the corresponding 12 points of that P_7. An example of this (with $k = 3; S = 4$) is shown in figure 3 to help with understanding. Formally, we have a set A of S pairs in P_k, and a set B of 12 pairs in P_7, we then make set

$$A' = \{(7a+b, 7c+d) \mid 0 \le a, c < k;\ 0 \le b, d < 7;\ (a,c) \in A;\ (b,d) \in B\}$$

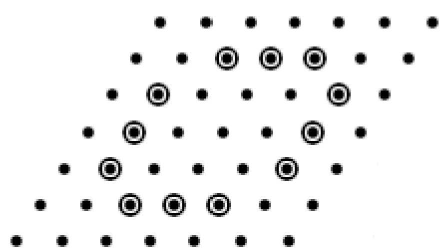

Figure 2: Maximum size equilateral-free set in P_7

Figure 3: Method for extending equilateral-free sets into larger ones

The reason A^0 will not contain any equilateral triangles is as follows. The points of the plane contained in it are a subset of the points defined by the 12 equivalence classes of P_7, so the only possible equilateral triangles must be within the same equivalence class in P_7. That means they must differ by multiples of 7 in axis parallel directions. Taking all such points reduces it to a copy of the original P_n, which will either be completely empty, or a copy of S, which is

$$\begin{cases} 2 & \text{if } 3 \text{ is a nonzero square in } F \\ 1 & \text{if } 3 \equiv 0, \text{ (so if } F \text{ has characteristic } 3) \\ 0 & \text{otherwise} \end{cases}$$

We will first solve to determine exactly what these x_3 are, if they exist.

Lemma 1.1 For any distinct x_1, x_2, x_3 in $\mathbb{Z}_p$ with $p > 2$, they form an equilateral triangle if and only if $x_3 = x_1 + R(x_2 - x_1)$ where

$$R = \begin{pmatrix} 1/2 & \mp\sqrt{3}/2 \\ \pm\sqrt{3}/2 & 1/2 \end{pmatrix}$$

Proof: The intuition behind R is that it represents a rotation of 60 degrees that will be applied to x_2 clockwise or counter-clockwise about x_1.
First, we can show that if $\sqrt{3} \neq 0$, the two solutions are not equal. This is because

$$\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}(x_2 - x_1) = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}(x_2 - x_1)$$

means that

$$\begin{pmatrix} 0 & -\sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}(x_2 - x_1) = 0$$

which means that either $x_2 - x_1 = 0$ which isn't allowed as they are distinct, or the matrix has nonzero kernel, which would mean its determinant is zero, so $3 = 0$, which is a contradiction.

Therefore, taking $x_3 = x_1 + R(x_2 - x_1)$ gives two distinct solutions if $\sqrt{3} \neq 0$ exists, one solution if $\sqrt{3} = 0$, and none if $\sqrt{3}$ does not exist. We know this is the total amount of solutions that will extend the segment (x_1, x_2) into an equilateral triangle, so if we show that these solutions do make an equilateral triangle, we will also have shown that no other solutions exist.

The rest is just computation. Letting $x_1 = (a, b)$, $x_2 = (c, d)$, so then $x_3 = (\frac{1}{2}(a + c) \mp \frac{\sqrt{3}}{2}(d - b); \frac{1}{2}(b + d) \pm \frac{\sqrt{3}}{2}(c - a))$, we can find the distance

$$||x_3 - x_1|| = (\frac{1}{2}(c - a) \mp \frac{\sqrt{3}}{2}(d - b))^2 + (\frac{1}{2}(d - b) \pm \frac{\sqrt{3}}{2}(c - a))^2$$

$$= (\frac{1}{4} + \frac{3}{4})(c - a)^2 + (\mp\frac{\sqrt{3}}{2} \pm \frac{\sqrt{3}}{2})(c - a)(d - b) + (\frac{1}{4} + \frac{3}{4})(d - b)^2$$

$$= (c - a)^2 + (d - b)^2 = ||x_2 - x_1||$$

and following similarly we also get

$$||x_3 - x_2|| = (\frac{1}{4} + \frac{3}{4})(c - a)^2 + (\mp\frac{\sqrt{3}}{2} \pm \frac{\sqrt{3}}{2})(c - a)(d - b) + (\frac{1}{4} + \frac{3}{4})(d - b)^2$$

$$= (c \quad a)^2 + (d \quad b)^2 = jj x_2 \quad x_1 jj$$

```
struct arr{ //struct used in order to be able to pass by val
        int ans[7][7];
};

void solve(arr ans, vector<pair<int, int>> marked, int x, int y, int tot){
```

```
for (int i=0; i<n; i++) f
        for (int j=0; j<n; j++) f
                ans.ans[i][j] = 0;
        g
g
vector<pair<int , int>> marked;
```