1.1. **Some Preparations.** A real-valued function $f : U \to \mathbb{R}$ defined on an open subset of $\mathbb{R}^n$ is **smooth** or C^∞ at a point p if its partial derivatives of all orders exists at p. Similarly, a function $f : U \to \mathbb{R}^m, f = (f_1, \ldots, f_m)$ defined on an open subset of $\mathbb{R}^n$ is **smooth** or C^∞ at a point p if each real-valued f_i is smooth at p. We say f is smooth on U if it is smooth at any point in U.

Fix $p \in \mathbb{R}^n$. We define S_p to be the set of all pairs (f, U) where U is a neighborhood[1] of p and $f : U \to \mathbb{R}$ a smooth function defined on U. We define an equivalence relation on S_p by $(f, U) \sim (g, V)$ iff f, g agree on some neighborhood of p, i.e. $f|_W = g|_W$ for some neighborhood $W \subseteq U \cap V$ of p. The family of equivalence classes, denoted $C_p^\infty := S_p/\sim$, is called **the germs of C^∞ functions at** p. Conventionally, we just say $f \in C_p^\infty$.

For any vector $v \in \mathbb{R}^n$, let D_v denote the **directional derivative at** p, so for $f \in C_p^\infty$, $D_v f = v \cdot (Df)_p$ where $(Df)_p = (\frac{\partial f}{\partial x^1}|_p, \ldots, \frac{\partial f}{\partial x^n}|_p)$[2]. In particular, note $D_v(fg) = g(p)(D_v f) + f(p)(D_v g)$. This is called the **Leibniz rule**. We say a linear map $D : C_p^\infty \to \mathbb{R}$ satisfying the **Leibniz rule** is a **derivation at** p or a **point-derivation** of C_p^∞.jf pis25 6.973838ibniz rul14 951 [(smo) -32 (oth)]TJ/F20 9.9626 Tf 4

Example 1.3. If M, N are manifolds, then $M \times N$ is also a manifold, with atlas $\{(U_i \times V_j, \phi_i \times \psi_j)\}$ where $\{(U_i, \phi_i)\}, \{(V_j, \psi_j)\}$ are atlases of M, N.

A Lie group is a manifold G with a group structure s.t. the multiplication map $G \times G \to G$ and the inversion map $G \to G$ are both *smooth*.

Let M, N be smooth manifolds. A function $f : N \to M$ is **smooth** if $\psi \circ f \circ \phi^{-1}$ is smooth for any chart (U, ϕ) on N and any chart (V, ψ) on M. It follows that charts on an n-dimensional manifold M are diffeomorphisms (between open subsets of M and $\mathbb{R}^n$), and diffeomorphisms (between open subsets of M and $\mathbb{R}^n$) are also charts.

Given a chart (U, ϕ) on M of dimension n, usually we write $\phi = (x^1, ..., x^n)$. The standard coordinates on $\mathbb{R}^n$ is denoted by $(r^1, ..., r^n)$, so $x^i = r^i \circ \phi$. Let $f : M \to \mathbb{R}$ be a smooth function. For $p \in U$, we define the **partial derivative** $\frac{\partial f}{\partial x^i}$ of f w.r.t. x^i to be

$$\frac{\partial f}{\partial x^i}|_p := \frac{\partial (f \circ \phi^{-1})}{\partial r^i}|_{\phi(p)}$$

Let $F : N \to M$ be a smooth map and $(U, \phi = x^1, ..., x^n), (V, \psi = y^1, ..., y^m)$ be charts on N, M respectively. We denote by $F^i := y^i \circ F$ the i-th component of F in the chart (V, ψ). Then the matrix

$$[\frac{\partial F^i}{\partial x^j}]$$

is called the **Jacobian matrix** of F relative to the charts $(U, \phi), (V, \psi)$. If M, N have the same dimension, the determinant of the Jacobian matrix is the **Jacobian determinant**.

Theorem 1.4. *Let* $F : N \to M$

basis for the tangent space T_pM. Relative to such bases, the differential of a smooth map $F : N \to M$ at $p \in N$ is represented by the Jacobian matrix $[\frac{\partial F^i}{\partial x^j}(p)]$.

A smooth map $F : N \to M$ is an **immersion** (resp. **submersion**) at p if its differential is injective (resp. surjective). A **regular value** is a point c in M s.t. F is a submersion at all points in $F^{-1}(c)$.

A subset S of a manifold N of dimension n is a **regular submanifold** of dimension k if for any $p \in S$, there is some chart $(U; x^1, \ldots, x^n)$ about p U, there is some

We have the following nice result for Lie groups. The term "left invariant" will be defined later when we use it.

Theorem 1.7. *A left invariant vector field on a Lie group generates a global flow.*

In particular, for any left invariant vector field X on a Lie group G and for any $g \in G$, the unique maximal integral curve of X starting at g is defined for all $t \in \mathbb{R}$.

Given a smooth vector field X and a real-valued smooth function f, Xf is also a smooth function where $(Xf)(p) := X_p f$. For vector fields X, Y, we define their **Lie bracket** $[X, Y]$ to be the vector field s.t. $[X, Y]_p f := X_p(Yf) - Y_p(Xf)$.

With these preparations, we now proceed to our discussion on Lie groups.

2. Lie Groups: Definitions and Examples

The roots of Lie theory can be traced back to the early 19th century, when mathematicians such as Evariste Galois and Niels Henrik Abel were working on the theory of equations and the solvability of algebraic equations by radicals. Galois in particular used group theory to study polynomial equations.

Inspired by Galois, Sophus Lie, a Norwegian mathematician, began working on the theory of continuous symmetry groups of differential equations, and showed how to use them to study differential equations.

Lie's work on Lie groups and Lie algebras was continued by a number of mathematicians in the early 20th century, including Elie Cartan, Wilhelm Killing, and Hermann Weyl. They developed the theory of semisimple Lie algebras and their representation theory, which has become a central topic in Lie theory.

We will review the definition of Lie groups and discuss several examples. At the end of the section, we will prove two lemmas about Lie groups, which will be used later.

2.1. Definition of Lie Groups.

We recall the definition of Lie groups.

Definition 2.1. A **Lie group** is a group and meanwhile a smooth manifold s.t. the group operation

$$\times : G \times G \to G, (g, h) \mapsto gh$$

and inversion

$$: G \to G, g \mapsto g^{-1}$$

are both smooth maps.

A **map** between two Lie groups G and H (or a Lie group map) is a map $: G \to H$ that is a group homomorphism and smooth.

Recall that there are two types of submanifolds - regular and immersed. We define a **Lie subgroup** of a Lie group G to be a subgroup that is also a regular submanifolds; and we define an **immersed subgroup** of G to be the image of an injective immersion of Lie groups $: H \to G$

Definition 2.2. A **representation** of a Lie group G is a morphism from G to $GL(V)$.

We will study the representation theory of Lie groups and Lie algebras. Before diving into this subject, I want to quickly mention its relation with representation theory of finite groups.

In representation theory of finite groups, we study how finite groups can act linearly on vector spaces, or, equivalently, how they can be mapped to $GL(V)$, the group of automorphisms of a vector space V (usually V is either real or complex). To study representation of Lie groups and Lie algebras is much more complicated, as they come with a topology and additionally a smooth structure. But the finite group case is still helpful when studying Lie groups and Lie algebras - many of the ideas used in representation theory of finite groups can be applied to Lie groups and Lie algebras. In fact, [1], the main reference of this paper, begins by spending six chapters on representation theory of finite groups and then goes on to discuss Lie groups and Lie algebras.

2.2. Examples.

Many subgroups of $GL_n\mathbb{R}$ are also Lie groups. For example, we have:

Example 2.3. The subgroup of $GL_n\mathbb{R}$ consisting of matrices with positive determinants, $GL_n^+\mathbb{R}$, is an open subset and actually connected [3], so it is the connected component of the identity matrix in $GL_n\mathbb{R}$.

Example 2.4. The special linear group $SL_n\mathbb{R}$ of $n \times n$ real matrices with determinant 1 is a connected Lie subgroup of codimension 1 by regular level set theorem.

Example 2.5. The group B_n of upper triangular matrices is a Lie subgroup (of $M_n\mathbb{R}$) of codimension $\frac{n(n-1)}{2}$. Similarly the group of invertible upper triangular matrices (i.e. nonzero on the diagonal) is a Lie subgroup of $GL_n\mathbb{R}$ of the same codimension.

Example 2.6. The group $\mathrm{Sym}_n\mathbb{R}$ of symmetric matrices is a Lie subgroup (of $M_n\mathbb{R}$) of codimension $\frac{n(n-1)}{2}$. This can be seen by noting that it is the zero set of the submersion f :

We will show later in Example 4.8 that their tangent spaces at the identity - their Lie algebras - have dimension $\frac{n(n-1)}{2}$ without knowing the dimension of the Lie groups.

Example 2.8. There are complex manifolds and hence complex Lie groups. For example $GL_n\mathbb{C}, U(n), SU(n)$, etc. Complex Lie groups are naturally also real Lie groups; their (complex) Lie algebras also naturally real. In the context of this paper, we only care about real ones.

In the representation theory of Lie groups, once we establish the correspondence between Lie groups and Lie algebras, one approach is to focus our attention on classification of Lie algebras, and complex Lie algebras are easier to classify, which we will not be able to discuss here. In fact, simple complex Lie algebras are completely classified.

2.3. Two Lemmas. Though with a more complicated structure, in some sense, Lie groups become a lot "cuter" than groups. Here are two interesting lemmas, which will be helpful later. We prove them under the Lie group setting, but the same proofs will work for topological groups in general.

Lemma 2.9. *Let G be a connected Lie group, and $U \subseteq G$ any neighborhood of the identity. Then U generates G.*

Proof. Let $U^{-1} = \{g^{-1} : g \in U\}$ and $V = U \cap U^{-1}$. Note U^{-1} is open as $g \mapsto g^{-1}$ is a diffeomorphism and hence a homeomorphism, so V is open. The purpose of this construction is that we now have $V^{-1} = V$. Consider $G' = \bigcup_{n\geq 1} V^n$. We show that G' is clopen (closed and open). Hence $G' = G$ and U generates G.

For any $x \in G$, xV is open as $g \mapsto xg$ is a diffeomorphism. Hence $V^n = \bigcup_{x\in V}$

3. Covering Spaces and Isogeny Class

One helpful concept in studying Lie groups is the isogeny class. To introduce it, we need to know covering spaces, which is itself a topic rich enough to write a paper on. We will rst review some main de nitions and properties of covering spaces, but will not prove them here. A detailed discussion on covering spaces can be found in any standard text in algebraic topology. Then we will prove two theorems regarding coverings of Lie groups, and nally we introduce the isogeny class.

We will see fundamental groups a few times in this section. Readers can also nd them in any text in algebraic topology if they are not familiar with this concept.

3.1. Covering Spaces. Let $p : E \to B$ be a map. For any $b \in B$, we call $p^{-1}(b)$ the ber of b. If we have maps $p_1 : E$

b

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In fact, covering spaces are much nicer than this that they have the *homotopy lifting property* (HLP) for all spaces, and hence a covering is a *fibration*. The path lifting property is the same as to say a covering has the HLP for a single point.

Theorem 3.5. *Let $p : E \to B$ be a covering. Suppose Z is path-connected and locally path-connected. Let $f : (Z, z) \to (B, p(x))$ be a map. Then there exists a lifting $\Phi : Z \to E$ of f with $\Phi(z) = x$ iff $f_* \pi_1(Z, z) \subseteq p_* \pi_1(E, x)$.*

$$\begin{array}{ccc} & & (E, x) \\ & \nearrow & \downarrow p \\ (Z, z) & \xrightarrow{f} & (B, p(x)) \end{array}$$

Note liftings in Proposition 3.4 and Theorem 3.5 are unique by Proposition 3.3.

We call a covering $p : E \to B$ a **universal covering** if E is simply connected.

Suppose B is path-connected, locally path-connected and semi-locally simply connected. Then it is guaranteed that a universal covering of B exists. Suppose $p : (E, x) \to (B, b)$ is a universal cover over B. Since B is locally path-connected and a covering is a local homeomorphism, the total space of a covering over B is also locally path-connected. Therefore, for any covering $p' : (E', x') \to (B, b)$, we can apply Theorem 3.5 and lift p to a (unique) pointed map $f : (E, x) \to (E', x')$ s.t. $p' \circ f = p$. When E' is connected, f is also a covering map,

We list some important properties but will not prove them.

Proposition 3.6. *For any B path-connected, locally-path-connected and semi-locally simply connected,*

1. *The universal covering $p : E \to B$ is unique up to isomorphisms.*
2. *The fibers of the universal cover p over B are (set) isomorphic to $\pi_1(B)$.*
3. *The group $\mathrm{Aut}(p) = \{f : E \to E : \text{is an automorphism and } p \circ = pg\}$, called deck transformations, is isomorphic to $\pi_1(B)$*
4. *Any map between connected coverings of B is also a covering map. In particular, any connected covering of B is also covered by the universal cover E.*

3.2. **Coverings of Lie Groups.** In this section, we will prove two theorems regarding coverings of Lie groups, and then introduce isogeny classes. We will first prove Theorem 3.9 that the covering of a Lie group is a Lie group. Before that, let us consider coverings of a manifold.

Recall a manifold is Hausdorff, second-countable and locally Euclidean. Any manifold is locally path-connected and locally simply connected (which implies semi-locally simply connected) as any open subset of $\mathbb{R}^n$ is. Thus a connected manifold M has a universal covering.

Let $p : E \to M$ be the universal covering of a connected manifold M. A natural question to ask is that is E also a manifold? The answer is yes. If $x, y \in E$ are distinct points with $p(x) \neq p(y)$, then since M is Hausdorff, we can take pullbacks of disjoint neigh2mq1(l)-389 0 Td [(.)]Toiny contained in $p(U)$ that is homeomorphic to an open subset of $\mathbb{R}^n$. Hence we get a neighborhood of x (contained in U, so homeomorphic to its image under p) that is homeomorphic to an open subset of $\mathbb{R}^n$. These are the charts on E induced by the

Remark 3.8. In the remaining of this section, we will see the term “smooth covering map” often. By using this term we are just specifying that the differentiable structure on the total space is the one induced by the covering map - what we naturally assume.

Dropping “smooth” and simply using “covering map” are actually harmless in our case. This is because given the topology and the group structure, the differentiable structure that will make the group a Lie group is unique. For example, in the next theorem, H is of course given a topology, and the uniqueness of the group structure will be shown at the very beginning, so the Lie group structure (if exists) must be unique.

We have the following result for Lie groups, analogous to Proposition 3.7:

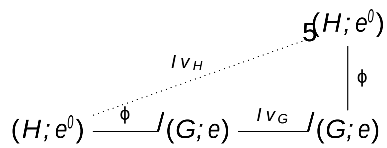
Theorem 3.9. *Let G be a Lie group, H a connected manifold, and $\phi : H \to G$ a covering map. Let $e' \in H$ be an element s.t $\phi(e') = e$ where e is the identity in G. Then there is a unique Lie group structure on H s.t. e' is the identity and ϕ is a map of Lie groups and a smooth covering map; and the kernel of ϕ lies in the center of H.*

Proof. To construct a group structure, consider the diagram

$$\begin{array}{ccccc} & & & & (H; e') \\ & & & \overset{\mu}{\nearrow} & \downarrow \phi \\ (H \times H; (e', e')) & \xrightarrow{\phi \times \phi} & (G \times G; (e, e)) & \xrightarrow{m} & (G; e) \end{array}$$

where m is the multiplication of G and $H \times H$ is path-connected and locally path-connected. We show that $m_*(\phi \times \phi)_*\pi_1(H \times H; (e', e'))$

Let the differentiable structure on H be the one induced by the covering map ϕ. Next we want to show that μ and the inversion map on H are smooth. Since $m \circ (\phi \times \phi)$ is smooth and ϕ a local diffeomorphism, μ is smooth. Let $Iv_G : G \to G$ be the inversion map $g \mapsto g^{-1}$. Note $Iv_G \circ \phi$ is also a covering over G, so it lifts to a unique map $Iv_H : H \to H$ that maps $e' \mapsto e'$.



Then the map $m \circ (\phi \times \phi) \circ (Id \times Iv_H) : H \to G$ is trivial. Its unique lifting, $\mu \circ (Id \times Iv_H)$, maps $e' \mapsto e'$ and maps everything into the (discrete) fiber over e. Hence $\mu \circ (Id \times Iv_H)$ must be trivial, meaning Iv_H is indeed the inversion map. Since $Iv_G \circ \phi$ is smooth and ϕ a local diffeomorphism, Iv_H is smooth.

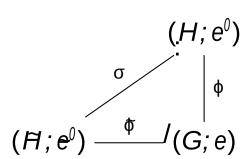
We have proven H is a Lie group and ϕ a Lie group map (and a smooth covering map). For the last statement, note that $\ker \phi$ is the fiber of e, which is a discrete subspace. By Lemma 2.10, $\ker \phi$ lies in the center of H.

Theorem 3.9 shows that connected coverings of a Lie group are also Lie groups. Furthermore, the universal cover of a Lie group is unique up to Lie group isomorphisms:

Proposition 3.10. *Let G be a connected Lie group. Let $\phi : H \to G, \tilde{\phi} : \tilde{H} \to G$ be connected coverings, where $\phi, \tilde{\phi}$ are Lie group maps and smooth covering maps. Suppose $\tilde{H}$ simply connected. Then there is a unique lifting $\sigma : \tilde{H} \to H$ mapping identity $\tilde{e}'$ to identity e' that is a Lie group map and a smooth covering map.*

In particular, this implies: The universal covering of a Lie group is unique in the sense that if G has universal coverings $\phi : H \to G, \tilde{\phi} : \tilde{H} \to G$ where $\phi, \tilde{\phi}$ are Lie group maps and smooth covering maps, then there is a Lie group isomorphism $\sigma : \tilde{H} \to H$ s.t. $\phi \circ \sigma = \tilde{\phi}$.

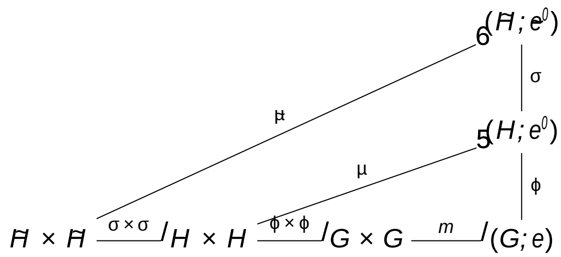
Proof. Let $(H; e'), (\tilde{H}; \tilde{e}')$ be two connected coverings of connected Lie group $(G; e)$, with $\phi, \tilde{\phi}$ being Lie group maps and smooth covering maps. Suppose $\tilde{H}$ simply connected. By Theorem 3.5, there is a unique map $\sigma : (\tilde{H}; \tilde{e}') \to (H; e')$ s.t.



commutes. By Proposition 3.6, σ is a covering map. We want to show σ is a smooth covering map and a group homomorphism.

We can take an open set U of G that is evenly covered by both ϕ and $\tilde{\phi}$ where each sheets are mapped diffeomorphically to U under $\phi, \tilde{\phi}$. Taking intersections if necessary, we can assume U is a chart of G and is path-connected. Note σ maps a sheet homeomorphically to another sheet. Since ϕ and $\tilde{\phi}$ are diffeomorphisms on these sheets, we conclude σ is diffeomorphic when restricted to one sheet. This shows that σ is a smooth covering map.

To see is a group homomorphism, we consider the diagram



where *m*; ; ~ are multiplication maps. Note the inner and outer triangles both commute as ; ~ = ∘ are both homomorphisms. Hence ∘ ~; ∘ (×) are both lifts of the map $m \circ ($ × $) \circ ($ ×

the choice of sheet U^0 does not matter, as other sheets are $gU^0; g$ 2 , which is di eomorphic to U^0. Under such charts, maps U^0 di eomorphically to U.

Hence G is a smooth manifold and a Lie group[6]

4.1. Definition of Lie Algebras. In th... we wi... art by introducing the *bracket operation*, and then introduce the ... ni... of Lie algebras.

Let G be a Lie group. For any $g \in G$, ... he co... gation map

$$\psi_g : G \to G, \ \mapsto g \ ...^{-1}$$

which is a Lie group automorphism. Note ... and ... We set

$$\mathrm{Ad}(g) = (\ ...$$

for all $X \in T_eG$, i.e.

$$d\ _e(\mathrm{ad}(X)(Y)) = \mathrm{ad}(d\ _e(X))(d\ _e(Y))$$

or, equivalently,

$$d\ _e([X; Y]) = [d\ _e(X); d\ _e(Y)]$$

This proves one direction of the Second Principle.

All above could be fairly confusing. In Example 4.5 we will see why we define the bracket in this way more explicitly. But now let us look at two important properties of the bracket operation.

Proposition 4.3. *With the notations as in the above discussion, we have*

1. *The bracket operation is skew-commutative, meaning* $[X; X] = 0$ *for all* $X \in T_eG$.
2. *The bracket operation satisfies the Jacobi identity:*

$$[X; [Y; Z]] + [Y; [Z; X]] + [Z; [X; Y]] = 0$$

Proof. For the first one, we use the fact that any $X \in T_eG$ is the derivative of a Lie group map $\ : \mathbb{R} \to G$ at 0 (we will prove this fact later). Note the Ad map on $\mathbb{R}$ is constant, so ad is the zero map for $\mathbb{R}$. Hence $[X; X] = [d\ _0(1); d\ _0(1)] = d\ _0[1; 1] = 0$.

For the second one, note that $\mathrm{ad}([X; Y]) = [\mathrm{ad}(X); \mathrm{ad}(Y)]$, as $\mathrm{ad} : T_eG \to \mathrm{End}(T_eG)$ is the differential of Ad at the identity. Using the fact that bracket operation for $GL(V)$ is $[M_1; M_2] = M_1M_2 - M_2M_1$ (see Example 4.5), we get

$$[[X; Y]; Z$$

For any $X, Y \in \mathfrak{g}$, let $\gamma : I \to G$ be an arc starting at e with tangent vector $\gamma'(0) = X$. Then our definition of $[X, Y]$ is that

$$[X, Y] = \mathrm{ad}(X)(Y) = \frac{d}{dt}\Big|_{t=0}(\mathrm{Ad}(\gamma(t))(Y))$$

Applying the product rule to $\mathrm{Ad}(\gamma(t))(Y) = \gamma(t)Y\gamma(t)^{-1}$[7], this yields

$$\begin{aligned} &= \gamma'(0)Y\gamma(0) + \gamma(0)Y(-\gamma(0)^{-1}\gamma'(0)\gamma(0)^{-1}) \\ &= XY - YX \end{aligned}$$

which explains the bracket notation.

In general, when a Lie group is a subgroup of $GL_n\mathbb{R}$, its Lie algebra is naturally embedded in $\mathfrak{gl}_n\mathbb{R}$ via the differential of the inclusion $\iota : G \hookrightarrow GL_n\mathbb{R}$; since bracket is preserved by this differential, the bracket operation on $\mathfrak{g} = T_eG$ coincides with the matrix bracket, i.e. the commutator.

Definition 4.6. **A representation of Lie algebra** $\mathfrak{g}$ on a vector space V is a map of Lie algebras

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V) = \mathrm{End}(V)$$

i.e., a linear map s.t. $\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$.

Viewing ρ as an action of $\mathfrak{g}$ on V, we have

$$[X, Y]v = X(Yv) - Y(Xv)$$

for all $v \in V$.

By the First Principle a representation $\rho : G \to GL_n\mathbb{R}$ of a connected Lie group G is completely determined by the representation of its Lie algebra $\rho : \mathfrak{g} \to \mathfrak{gl}_n\mathbb{R}$ given by the differential of ρ. By the Second Principle the representations of a simply connected Lie group are in 1-1 correspondence with the representations of its Lie algebra.

Example 4.7. Consider the special linear group $SL_n\mathbb{R}$. Let $A(t)$ be an arc in $SL_n\mathbb{R}$ starting at $A(0) = I$ and with tangent vector $A'(0) = X$ at $t = 0$. Then

$$\det A(t) = \sum_{\sigma \in S_n} \mathrm{sgn}\,\sigma \prod_{i=1} A(t)_{(i,\sigma(i))} = 1$$

for all t, wh[illegible]

Example 4.8. Consider the orthogonal group $O_n\mathbb{R}$, as the group of automorphisms on $V = \mathbb{R}^n$ preserving the inner product. Let $A(t)$ be an arc in $O_n\mathbb{R}$ starting at $A(0) = I$ and with tangent vector $A'(0) = X$ at $t = 0$. We know $A(t)^T A(t) = I$ for all t. Hence, taking derivatives and evaluating at $t = 0$, we get

$$X^T + X = 0$$

i.e. elements in $\mathfrak{o}_n\mathbb{R}$ are skew-symmetric matrices.

We will show later using *exponential map* that any skew-symmetric matrix lies in $\mathfrak{o}_n\mathbb{R}$. Thus $\mathfrak{o}_n\mathbb{R}$ is precisely the space of skew-symmetric matrices, which clearly has dimension $\frac{n(n-1)}{2}$. Since $SO_n\mathbb{R}$ is an open subset of $O_n\mathbb{R}$, $\mathfrak{so}_n\mathbb{R} = \mathfrak{o}_n\mathbb{R}$.

We close this section by stating a deep result about Lie algebras, which we will use later. Interested readers can nd the proofs in Chapter 3.17 of [6] and Appendix E of [1].

Theorem 4.9. *(Ado's Theorem) Every nite-dimensional real Lie algebra admits a faithful nite-dimensional representation.*

Hence any nite-dimensional Lie algebra is isomorphic to a Lie subalgebra of $\mathfrak{gl}_n\mathbb{R}$ for some n.

4.3. **The Exponential Map.** The exponential map is an essential topic in studying the relationship between a Lie group and its Lie algebra. Precisely, it gives a map from a Lie algebra to its Lie group.

Let G be a Lie group. We will rst associate 1-1 correspondences between the following sets:

(1) The Lie algebra $\mathfrak{g}$ of Lie group G, i.e. the tangent space at the identity.
(2) The *left invariant vector elds* on G.
(3) The *one-parameter subgroups* of G.

De nition 4.10. Let X be a (smooth) vector eld on G. We say X is **left invariant** if,

$(DL_x)_g X_g = X_{xg};$ for all $x; g$ 2 9.9626 Tf 83.1 0 Td 9626 [(g)]TJ/F2387 -Icon where the bracket is induced by the Lie bracket on the vector elds (see Theorem 20.27 in [4]).

De nition 4.12. A **one-parameter subgroup** of G is a Lie group homomorphism $: \mathbb{R} ! G$ where $\mathbb{R}$ is considered as a Lie group under addition.

First we show that a one-parameter subgroup is uniquely determined by the tangent vector $^0(0)$ at the identity. Let $X = {}^0(0)$ and let X also denote the left invariant vector eld generated by $^0(0)$ by abusing the use of notations. Since $(s + t) = (s)(t)$, di erentiating on both sides w.r.t. t and evaluating at $t = 0$ results in $^0(s) = (DL_{\gamma(s)})_e {}^0(0) = (DL_{\gamma(s)})_e X_e = X_{\gamma(s)}$. Thus is an integral curve of X starting at the identity (and a maximal one), and is uniquely determined by $^0(0)$.

Next we show that given a left invariant vector eld X on G, we can generate a unique one-parameter subgroup. Together with the previous paragraph, we conclude one-parameter subgroups are isomorphic to left invariant vector elds.

By Theorem 1.7, there is a unique maximal integral curve of X starting at the identity that is de ned on all $t\ 2\ \mathbb{R}$, i.e. a smooth curve $: \mathbb{R} ! G$, with $(0) = e$ and $^0(t) = X_{\phi(t)}$. We will show that is a one-parameter subgroup. That means we need to show is a homomorphism, i.e. $(s + t) = (s)(t)$.

Let s be xed and de ne two arcs $;$ by

$$(t) = (s)(t); \quad (t) = (s + t); t\ 2\ \mathbb{R}$$

Then $(t) = (L_{\phi(s)} \circ)(t)$, so $^0(t) = (DL_{\phi(s)})_{\phi(t)} {}^0(t) = X_{\phi(s)\phi(t)} = X_{\alpha(t)}$ as X is left invariant. On the other hand, $^0(t) = {}^0(s + t) = X_{\phi(s+t)} = X_{\beta(t)}$. Hence $;$ are integral curves of X with the same starting point. Thus $(t) = (t)$ for all t, and $(s + t) = (s)(t)$. We conclude is a one-parameter subgroup and denote it by $_X$.

To sum up, the following are identi ed with each other:

(1) The tangent vector $X\ 2$ g;
(2) The left invariant vector eld generated by $X\ 2$ g;
(3) The one-parameter subgroup with tangent vector X at 0 (or generated by X), which is the same as
(4) The maximal integral curve of vector eld X starting at e.

We are now ready to de ne the exponential map.

De nition 4.13. We de ne the **exponential map**

expG; X 7! X(1)]TJ/F20 9.9626 Tf 5.858 0 Td6.32315 0 T [(i)-278(=)]TJ/F23 9.9626 Tf 17.158 0

(1) *The exponential map is smooth.*
(2) *For any* $X \in \mathfrak{g}; s, t \in \mathbb{R}$, $\exp(s+t)X = \exp sX \exp tX$.
(3) *For any* $X \in \mathfrak{g}$, $(\exp X)^{-1} = \exp(-X)$.
(4) *The differential* $(D\exp)_0$ *is the identity map*[8].
(5) *The exponential map restricts to a diffeomorphism from some neighborhood of* $0 \in \mathfrak{g}$ *to a neighborhood of* $e \in G$.
(6) *If* $f : G \to H$ *is a Lie group map, then the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{f} & \mathfrak{h} \\ {\scriptstyle \exp}\big\downarrow & & \big\downarrow{\scriptstyle \exp} \\ G & \xrightarrow[f]{} & H \end{array}$$

Proof. For any $X \in \mathfrak{g}$, let θ_X denote the (global) flow generated by (the left invariant vector field generated by) X. To show that $\exp : \mathfrak{g} \to G$ is smooth, we need to show that $\theta^e_X(1) = \exp X$ depends smoothly on X.

Define a vector field Θ on the product manifold $G \times \mathfrak{g}$ by by Proposition 4.14.

(5) follows immediately from (4) and Theorem 1.4.

(6): To show the diagram commutes, i.e. $f(\exp X) = \exp(f_* X)$ for all $X \in \mathfrak{g}$, we note that $f(\exp tX)$ is a one-parameter subgroup of H whose tangent vector at 0 is

$$\frac{d}{dt}|_{t=0}(f \exp tX) = f_*((\exp tX)'(0)) = f_* X$$

Hence $f(\exp X) = \exp(f_* X)$.

Example 4.16. Recall the matrix exponential. For a matrix $A \in M_n\mathbb{R}$, we define

$$e^A := \sum_{n=0}^{\infty} \frac{1}{n!}A^n = I + A + \frac{1}{2}A^2 + \cdots$$

which always converges and the result e^A is invertible with inverse e^{-A}. Let $\gamma : \mathbb{R} \to GL_n\mathbb{R}$ be the map $t \mapsto e^{tA}$. Then $\gamma'(0) = A$; i.e. γ is the one-parameter subgroup of $GL_n\mathbb{R}$ with tangent vector A at 0. We now see that the exp map we define previously coincides with the matrix exponential, which explains the name "exponential map".

Example 4.17. Let A be a skew-symmetric matrix, i.e. $A^T = -A$. Then

$$\exp(A)\exp(A)^T = \exp(A)\exp(A^T) = \exp(A)\exp(-A) = I$$

meaning that $\exp(A)$ is orthogonal. Hence $\exp(tA)$ is a one-parameter subgroup of $O(n)$, so $A = \exp(tA)'|_{t=0}$

satisfy $F(\exp(X)) = \exp(f(X))$. If everything is nice enough (e.g. supposing $\exp(X)\exp(Y) = \exp(Z)$ and $\exp(f(X))\exp(f(Y)) = \exp(f(Z))$), we would have

$$\begin{aligned} F(\exp(X)\exp(Y)) &= F(\exp(Z)) \\ &= \exp(f(Z)) \\ &= \exp(f(X))\exp(f(Z)) \\ &= F(\exp(X))F(\exp(Y)) \end{aligned}$$

which is what we want to make F a group homomorphism.

5. Lie Group-Lie Algebra Correspondence

With all things we have introduced, we will now prove our main results - the correspondence between Lie groups and Lie algebras. This correspondence consists of several crucial results, including the forementioned First Principle and Second Principle.

The First Principle follows easily from Property (5) and (6) of Proposition 4.15 and Lemma 2.9. If we are given the differential of a Lie group map $f : G \to H$, then, as exp is a local diffeomorphism at the identity and G is connected, f is completely determined by its differential.

Next we prove the following **correspondence between Lie subgroups and Lie subalgebras.** [9]

Theorem 5.1. *Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra. Then the subgroup of G generated by* $\exp(\mathfrak{h})$ *is an immersed subgroup H with tangent space $T_eH = \mathfrak{h}$.*

Proof. Note that the subgroup generated by $\exp(\mathfrak{h})$ is the same as the subgroup generated by $\exp(U)$ for any neighborhood U of the origin in $\mathfrak{h}$.

Let be a disk centered at the origin in $\mathfrak{g}$ on which exp is a diffeomorphism and the BCH formula holds. I.e., for $X, Y \in$, we have $\exp(X)\exp(Y) = \exp(Z)$ where Z can be represented by the convergent series:

$$Z = (X + Y) + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] - [Y, [X, Y]]) + \dots$$

Let $G_0 = \exp(\ )$, $H_0 = \exp(\ \cap \mathfrak{h})$. We show that the subgroup H of G generated by H_0 is an immersed subgroup of G with tangent space $T_eH = \mathfrak{h}$.

Observe that, as is a disk, we have $G_0^{-1} = G_0$ and $H_0^{-1} = H_0$. Hence the subgroup H generated by H_0 is precisely

$$H = \cup_{n \geq 1} H_0^n$$

We put a topology on H. Let H_0 be open in H, which is naturally homeomorphic to $\ \cap \mathfrak{h}$ under exp. Any coset hH

a Lie group homomorphism is always of constant rank. Thus H is our desired immersed subgroup whose Lie algebra is (isomorphic to) $\mathfrak{h}$. [11]

Note that the connected component of H in Theorem 5.1 is the unique connected immersed subgroup of G with Lie algebra $\mathfrak{h}$. This is because any connected immersed subgroup with Lie algebra $\mathfrak{h}$ must contain $\exp(\mathfrak{h})$.

Recall that by Ado's Theorem, every finite-dimensional real Lie algebra is isomorphic to a Lie subalgebra of $\mathfrak{gl}_n\mathbb{R}$ for some n. Thus Theorem 5.1 tells us that every finite-dimensional Lie algebra is (isomorphic to) the Lie algebra of a Lie group and, by Theorem 3.9, of a simply connected Lie group. This is nowadays known as the **Lie's Third Theorem**:

Theorem 5.2. *Every finite-dimensional real Lie algebra is the Lie algebra of some simply connected Lie group.*

As a another consequence of Theorem 5.1, we are now ready to prove the Second Principle, also called **the homomorphism theorem**:

Theorem 5.3. *Let G and H be Lie groups, with G simply connected. A linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ is the differential of a Lie group map $\Phi : G \to H$ iff it preserves brackets, i.e. a Lie algebra map.*

Proof. Consider the product $G \times H$, whose Lie algebra is (isomorphic to) $\mathfrak{g} \oplus \mathfrak{h}$. Let $\mathfrak{j} \subset \mathfrak{g} \oplus \mathfrak{h}$ be the graph of the map φ; i.e. $\mathfrak{j} = \{(v, \varphi(v)) : v \in \mathfrak{g}\}$. First we show that $\mathfrak{j}$ is a Lie subalgebra. Clearly it is a subspace. We need to show that it is closed under bracket.

Note that the left invariant vector field on $G \times H$ generated by $(v, 0) \in \mathfrak{g} \oplus \mathfrak{h}$ is $X_{(g,h)} = (X_g, 0)$ where $X_g = (DL_g)_e v$, the left invariant vector field on G generated by v. (This is because left multiplication by bbbP[(v)-36(;)-1 [(h)]TJ/F20 9.9626 Tf 27.167 0 Td [(is)-453(is)-4

Corollary 5.4. *If $G; H$ are simply connected Lie groups with isomorphic Lie algebras, then $G; H$ are isomorphic.*

Proof. By Theorem 5.3 we have Lie group maps in both directions, and the differentials of their compositions are the identity maps on the Lie algebras. Thus their compositions must be the identity maps on the Lie groups by the First Principle.

The three theorems we introduced in this section are the main results of the **Lie group-Lie algebra correspondence**. We collect them as follows:

- (The Subgroup-Subalgebra Correspondence, done in Theorem 5.1) Let G be a Lie group, g its Lie algebra, and h g a Lie subalgebra. Then there is a unique immersed subgroup H of G with Lie algebra T_eH = h.
- (Lie's Third Theorem, done in Theorem 5.2) Every nite-dimensional real Lie algebra is the Lie algebra of some simply connected Lie group.
- (The Homomorphism Theorem, done in Theorem 5.3) Let G and H be Lie groups, with G simply connected. Any Lie algebra map : g ! h is the di erential of a unique Lie group map : G ! H.

Together they give **the Lie Correspondence**:

Theorem 5.5. *(The Lie Correspondence) There is a one-to-one correspondence between isomorphism classes of nite-dimensional Lie algebras and isomorphism classes of simply connected Lie groups, given by associating each simply connected Lie group with its Lie algebra.*

Proof. Injectivity is given by Corollary 5.4 and surjectivity by Theorem 5.2.

6. A Word on Representation of Lie Groups

At the end of this paper, let's take a quick look at how representation theory of Lie groups develop.

Recall that by the First Principle, a representation : G ! $GL_n\mathbb{R}$ of a connected Lie group is uniquely determined by the corresponding Lie algebra representation : g ! $\mathrm{gl}_n\mathbb{R}$. Together with the Lie group-Lie algebra correspondence, we reduce the representation theory of Lie groups to representation theory of Lie algebras,

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