TWO GEOMETRIC COMBINATORIAL PROBLEMS IN VECTOR SPACES OVER FINITE FIELDS

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Abstract. First, we show that the number of ordered right triangles with vertices in a subset E of the vector space $\mathbb{F}_q^2$ over the finite field $\mathbb{F}$

Often, we will ask how large a subset E of $\mathbb{F}_q^d$

Proposition 2. (Plancherel's Formula) *Let* $f, g : \mathbb{F}_q^d \to \mathbb{C}$*. Then*

$$\sum_{m \in \mathbb{F}_q^d} \hat{f}(m)\overline{\hat{g}(m)} = q^{-d} \sum_{x \in \mathbb{F}_q^d} f(x)\overline{g(x)}.$$

Proof.

$$\sum_{m \in \mathbb{F}_q^d} \hat{f}(m)\overline{\hat{g}(m)} = q^{-2d} \sum_{m \in \mathbb{F}_q^d} \sum_{x \in \mathbb{F}_q^d} f(y)\chi(-x \cdot$$

for z E. This yields at most $|E|^2$ combinations of values for y, z, and z. Then

where $|\theta| < 2^{1/2}$. It follows from direct computation that the first term exceeds the second if $|E| > 2^{\frac{1}{3}} q^{\frac{5}{3}}$. Now if $q \equiv 3 \pmod 4$, Lemma 2 instead yields

$$q(q-1)|E| \sum_{\substack{y,z,y',z' \in E \\ y+z=y'+z' \\ y\cdot z=y'\cdot z'}} 1 \le q(q-1)|E|^3(q+2) = (q^3+q^2-2q)|E|^3.$$

Hence the second term is bounded by $q^{\frac{3}{2}}|E|^{\frac{3}{2}}(1+o(1))$. Hence the first term exceeds the second when $q^{-1}|E|^3 > q^{\frac{3}{2}}|E|^{\frac{3}{2}}(1+o(1))$, i.e. when

$$|E| > q^{\frac{5}{3}}(1+o(1)).$$

This concludes the proof of Theorem 2. □

4. Discrepancies

4.1. **Statement of Results.** A hyperplane in $\mathbb{F}_q^d$ is a set of the form $\{x \in \mathbb{F}_q^d : x \cdot m = t$

To show that the map is onto, we recall that every $H \in H$ can be written as $\{x \in \mathbb{F}_q^d : x \cdot m = s\}$ for some $s \in \mathbb{F}_q$ and nonzero m in $\mathbb{F}_q^d$. Then there exist unique $v \in V(\mathbb{F}_q^d)$ and $\lambda \in \mathbb{F}_q \setminus \{0\}$ such that $m = \lambda v$. Then we write

$$\begin{aligned}\{x \in \mathbb{F}_q^d : x \cdot m = s\} &= \{x \in \mathbb{F}_q^d : x \cdot \lambda v = s\} \\ &= \{x \in \mathbb{F}_q^d : x \cdot v = \lambda^{-1} s\} \\ &= H_{v,\lambda^{-1}s}.\end{aligned}$$

□

Proof. We have by the above proposition,

$$
\begin{aligned}
\sum_{H \in \mathcal{H}} |E \cap H|^2 &= \sum_{\substack{v \in V(\mathbb{F}_q^d)\\ t \in \mathbb{F}_q}} |E \cap H_{v,t}|^2 \\
&= \sum_{\substack{v \in V(\mathbb{F}_q^d)\\ t \in \mathbb{F}_q}} \left| q^{-1} \sum_{x \in \mathbb{F}_q^d} E(x) \sum_{s \in \mathbb{F}_q} \chi(s(t - x \cdot v)) \right|^2 \\
&= q^{-2} \sum_{\substack{v \in V(\mathbb{F}_q^d)\\ t \in \mathbb{F}_q}} \sum_{\substack{x,x' \in \mathbb{F}_q^d\\ s,s' \in \mathbb{F}_q}} E(x)E(x')\chi((s - s')t)\chi((-sx + s'x') \cdot v) \\
&= q^{-1} \sum_{v \in V(\mathbb{F}_q^d)} \sum_{\substack{x,x' \in \mathbb{F}_q^d\\ s \in \mathbb{F}_q}} E(x)E(x')\chi(s(-x + x') \cdot v) \\
&= q^{2d-1} \sum_{\substack{v \in V(\mathbb{F}_q^d)\\ s \in \mathbb{F}_q}} \left| q^{-d} \sum_{x \in \mathbb{F}_q^d} E(x)\chi(-sx \cdot v) \right|^2 \\
&= q^{2d-1} \sum_{\substack{v \in V(\mathbb{F}_q^d)\\ s \in \mathbb{F}_q}} |\hat{E}(sv)|^2 \\
&= q^{2d-1} \sum_{\substack{v \in V(\mathbb{F}_q^d)\\ s \in \mathbb{F}_q^*}} |\hat{E}(sv)|^2 + q^{2d-1} \sum_{v \in V(\mathbb{F}_q^d)} |\hat{E}(0)|^2.
\end{aligned}
$$

Since $V(\mathbb{F}_q^d)$ is a direction set, we have

$$
= q^{2d-1} \sum_{m \in \mathbb{F}_q^d \setminus \{0\}} |\hat{E}(m)|^2 + q^{-1}|V(\mathbb{F}_q^d)||E|^2
$$

References

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[2] J. Pach, and P. Agarwal *Combinatorial geometry* Wiley-Interscience Series in Discrete Mathematics and O-0.2s