The Development of the Hardy Inequality

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Abstract

This paper is a discussion of the development of the Hardy Inequality. We detail the inequality in both the discrete and continuous cases, as well as notable work, by Hardy and other mathematicians at the time, that contributed to its development. Much of the content draws upon an article from The American Mathematical Monthly by Alois Kufner, Lech Maligranda and Lars-Erik Persson [8].

1 Introduction

1.1 Lebesgue Spaces

Before going on to discuss these inequalities, we will first briefly discuss the Lebesgue Spaces L^p and ℓ_p and related notation and terminology. We will be using the notation used in [1].

Given a measure space (X, S, μ) and $0 < p \leq \infty$, the Lebesgue Space L^p, or $L^p(X, S, \mu)$, is the set of S-measurable functions $f : X \to F$ such that $||f||_p < \infty$, where $||f||_p$ is the p-norm of f, and is defined as follows:

- if $0 < p < \infty$: $||f||_p = \left(\int |f|^p d\mu\right)^{\frac{1}{p}}$
- $||f||_\infty = \inf\{t > 0 : \mu(\{x \in X : |f(x)| > t\}) = 0\}$.

Moreover, when the measure μ is the counting measure on $\mathbb{Z}^+$, and $a = (a_1, a_2, \ldots)$ is a sequence in F and $0 < p < \infty$, then

$$||a||_p = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}}; \text{ and } ||a||_\infty = \sup\{|a_n| : k \in \mathbb{Z}^+\}$$

and we write ℓ_p in place of $L^p(\mu)$.

If we restrict p to the interval $[1, \infty]$ and write that $Z(\mu)$ be the set of S-measurable functions that vanish except on a set of order zero, and denote the quotient space $L^p(\mu)/Z(\mu)$ as $L_p(\mu)$, we construct a Banach space of p-integrable functions. In L_p we simply say that the p-norm of a function f, $||f||_p$, is equal to the norm of its representative in L^p. In the case of the counting measure, $L^p(\mu) = L_p(\mu) = \ell_p(\mu)$ as, with respect to this measure, the only set of measure zero is the empty set.

2 Motivation and Prior Results

Before detailing Hardy's proofs of the main results, we will address Hardy's motivation for beginning work toward these results, important theorems that will be of use to a reader of this paper, and Hardy's related results prior to his proof of his continuous and discrete inequalities.

2.1 Motivation: The Hilbert Inequality

In the article by Kufner, Maligranda, and Persson [8], they write that "it seems completely clear that Hardy's original motivation when he began the research that culminated in his discovery of the inequalities [(1) and (2)] was to prove (the weak form of) the Hilbert inequality." This inequality (and variants of it) pertains to sequences $\{a_m\}_{m}^{\infty}$ and $\{b_n\}_{m}^{\infty}$ of nonnegative real numbers such that $\sum_{m=1}^{\infty} a_m^2 < \infty$ and $\sum_{n=1}^{\infty} b_n^2 < \infty$, or in the notation we introduced earlier: $a, b \in \ell_2(\mu)$. In this context, we have the following variants of the Hilbert Inequality:

- Weak form: the double series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} \text{ converges} \tag{3}$$

- Typical (Strong) form: the inequality

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} \qquad \left(\sum_{m=1}^{\infty} a_m^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}} \tag{4}$$

m TypGener(y)zati+ $\sum_{n} \sum_{m=1} a_m b$

holds, and is a sharp constant.

2.2.3 Holder's Inequality

If $(X, \mathcal{S}, \mu)$ is a measure space, $1 \le p \le \infty$, and $f, h : X \to \mathbf{F}$ are $\mathcal{S}$-measurable, then

$$\|fh\|_1 \le \|f\|_p \|h\|_{p'} \tag{8}$$

2.2.4 Minkowski's Inequality

If $(X, \mathcal{S}, \mu)$ is a measure space, $1 \le p \le \infty$, and $f, h \in L^p(\mu)$ then

$$\|f$$

Theorem 2.4. let $a > 0$, $f(x); g(x)$ be non-negative and integrable on $(a; \infty)$, and denote $F(x) = \int_a^x f(t)dt$, $G(x) = \int_a^x g(t)dt$ The following hypotheses are equivalent:

a proof of the discrete inequality in the following form [8]:

Theorem 2.8 (Proved by Landau). If $p > 1$, $a_n \geq 0$, $\sum a_n^x$ is convergent, then

$$\sum_{n=1}^{N} \left(\frac{1}{n} \sum_{k=1}^{n} a_k \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{N} a_n^p$$

and the constant $\left(\frac{p}{p-1}\right)^p$ is sharp when $N = \infty$.

Additionally, in another letter, Landau drew Hardy's attention to the fact that, in his 1920 paper, Hardy had remarked that $(p/(p-1))^p$ was the best constant in the continuous case, without providing a proof. This exchange was addressed in Hardy's 1925 paper alongside such a proof.

3 Proof of the Hardy Inequality

We will now present Hardy's proofs of (1) and (2) as they were originally presented in his 1925 paper [7], albeit with slight modification for notational simplicity.

3.1 Continuous Case

Theorem 3.1. Suppose that $f(x) \geq 0$, $p > 1$, that f Lebesgue integrable over any finite interval $(0, X)$, and

$$F(x) = \int_0^x f(t)dt;$$

and that $f \in L^p(\mathbb{R}^+)$. Then

$$\int_0^\infty \left(\frac{F(x)}{x} \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f(x)^p dx.$$

Proof. By applying integration by parts and making use of the chain rule $\frac{d}{dx}F(x)^p = pF(x)^{p-1}f(x)$:

$$\begin{aligned}
\int_\epsilon^X \left(\frac{F(x)}{x} \right)^p dx &= -\frac{1}{p-1} \int_\epsilon^X F(x)^p \frac{d}{dx}(x^{1-p})dx \\
&= \frac{\epsilon^{1-p}}{p-1}F(\epsilon)^p - \frac{X^{1-p}}{p-1}F(X)^p + \frac{1}{p-1}\int_\epsilon^X x^{1-p}\frac{d}{dx}(F(x)^p)dx \\
&= \frac{\epsilon^{1-p}}{p-1}F(\epsilon)^p - \frac{X^{1-p}}{p-1}F(X)^p + \frac{p}{p-1}\int_\epsilon^X x^{1-p}F(x)^{p-1}f(x)dx \\
&\leq \frac{\epsilon^{1-p}}{p-1}F(\epsilon)^p + \frac{p}{p-1}\int_\epsilon^X x^{1-p}F(x)^{p-1}f(x)dx.
\end{aligned}$$

When $\xi \to 0$, by Holder's Inequality (8), we have that

$$F(\xi)^p = \left(\int_0^\xi f(t)dt\right)^p \le \int_0^\xi f(t)^p dt \left(\int_0^\xi dt\right)^{p-1} = o(\xi^{p-1}).$$

For $\epsilon > 0; \exists \delta$ so that $\forall \xi \le \delta$:

$$F(\xi)^p < (p-1)\,\epsilon\, ||f||_p^p\, \xi^{p-1}.$$

Also,

$$\int_\delta^x \left(\frac{F(x)}{x}\right)^{p-1} f(x)dx \le \left(\int_\delta^x f(x)^p dx\right)^{\frac{1}{p}} \left(\int_\delta^x \left(\frac{F(x)}{x}\right)^p dx\right)^{\frac{1}{p'}}$$

$$\le ||f||_p^p \left(\int_\delta^x \left(\frac{F(x)}{x}\right)^p dx\right)^{\frac{1}{p'}}$$

Denoting $W = \int_\delta^x \left(\frac{F(x)}{x}\right)^p dx$, this yields that:

$$W < \epsilon ||f||_p^p + \frac{p}{p-1} ||f||_p W^{1/p'}$$

or, equivalently,

$$\left(\frac{W}{||f||_p^p}\right)^{\frac{1}{p}} \le \frac{p}{p-1} \ldots$$

W p

Theorem 3.2. The inequality (2) is strict $((p=(p \quad 1))^p$ is the best possible constant).

Proof. To show that this is the best possible constant, we proceed as follows:

$$\text{let} \quad > 0; \quad f = \begin{cases} 0 & 0 \quad x < 1 \\ x^{\;(\;+\;)} & 1 < x \end{cases};$$

where $\quad = 1=p; \quad 0 < \quad < \frac{1}{2}(1 \qquad) < 1$

If $_{n\ 1}$ x $_n$, then

$$\frac{F(x)}{x} = \frac{{}_1a_1 + ::: + {}_{n\ 1}a_{n\ 1} + (x \quad {}_{n\ 1})a_n}{x};$$

which will decrease monotonically from $A_{n\ 1} = {}_{n\ 1}$ to $A_n = {}_n$ as x increases from $_{n\ 1}$ to $_n$. Thus

$$\frac{F(x)}{x} \quad \frac{A_n}{{}_n} \quad \text{when} \quad {}_{n\ 1} \quad x < {}_n:$$

Combining these observations with Theorem 3.1 and the earlier simpli cations that we made yields the desired result:

$$\sum_{n=1} {}_n \left(\frac{A_n}{{}_n}\right)^p \quad \int_0^1 \left(\frac{F(x)}{x}\right)^p \quad \frac{p}{p\quad 1}^p \int_0^1 f(x)^p dx = \frac{p}{p \quad 1} \sum_{n=1} {}_n a_n^p$$

□

4 Further Results

Having proven our main result, we will now proceed to explore alternative proofs and further results.

[redacted] Polya's Proof

[redacted]n that very same article, Hardy also shared an alternative proof of Theorem 3.1 which had been pointed out to him by George Polya in their correspondence. This proof, while beginning in the same way, makes a few notable simpli cations and thus avoids some of the more technical arguments in Hardy's proof. We present this alternative proof here:

Proof. Suppose that $0 < \quad < \quad < X$: Recall, from Hardy's proof, that:

$$\int^X \left(\frac{F(x)}{x}\right)^p dx \quad \frac{{}^{1\ p}}{p \quad 1} F(\ )^p + \frac{p}{p \quad 1} \int^X x^{1\ p} F(x)^{p\ 1} f(x) dx$$

By taking this inequality and replacing $F(x)$ with $F(x) \quad F(\ )$ and with , dropping the rst term (which is non-negative), and applying Holder's inequality (8) in the same way we do in [redacted]'s proof, this yields:

$$(\ ) \quad \int^X \left(\frac{F(x) \quad F(\ )}{x}\right)^p dx \quad \frac{p}{p \quad 1} \int^X \left(\frac{F(x) \quad F(\ )}{x}\right)^{p\ 1} f(x) dx$$

p

$$p \quad 1 \quad \int_0^1 f(x)^p::$$

As $f(x)$ is non-negative, $F(x)$ will be monotonically increasing. Thus, $F(x) - F(\epsilon)$ increases monotonically to $F(x)$ as $\epsilon \to 0$. Hence:

$$\int_{\epsilon}^{X} \left(\frac{F(x)}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f(x)^p dx;$$

which proves the theorem, as ϵ and X are arbitrary and we can apply this inequality for $\epsilon \to 0$ and $X \to \infty$. □

4.2 The Hardy Operator

One consequence of Hardy's Inequality, is that the discrete Hardy operator h and the continuous Hardy operator H, defined by:

$$h(a_n) = \left\{\frac{1}{n}\sum_{k=1}^{n} a_k\right\}; \quad Hf(x) = \frac{1}{x}\int_0^x f(t)dt;$$

map the spaces l_p and L_p ($p > 1$) into themselves, respectively. Moreover, each of these operators have norm $p' = \frac{p}{p-1}$.

Proof. First, we recall that L_p and l_p are Banach spaces. The norm of a linear map, T between Banach spaces V and W is given by $||T|| = \sup\{||Tf||_W : f \in V \text{ and } ||f||_V \leq 1\}$. Rephrasing Hardy's inequality in terms of the l_p and L_p norms and the continuous and discrete Hardy operators yields:

$$||h(\{a_n\})||_{l_p}^p \leq \left(\frac{p}{p-1}\right)^p ||a||_{l_p}^p$$
$$||Hf(x)||_{L_p}^p \leq \left(\frac{p}{p-1}\right)^p ||f(x)||_{L_p}^p:$$

By exponentiation of both equations by $1/p$, we arrive at:

$$||h(\{a_n\})||_{l_p} \leq \frac{p}{p-1}||a||_{l_p}$$
$$||Hf(x)||_{L_p} \leq \frac{p}{p-1}||f(x)||_{L_p};$$

where the constant $p' = \frac{p}{p-1}$ is the best constant (for any sequence or function). Paying special attention to the case of functions in L_p whose norms are less than or equal to 1, by the definition of the norm of a linear map and these inequalities, we immediately arrive at the desired result. □

4.3 Ingham's Proof of The Hardy Inequality

By making use of the Hardy operator and Minkowski's Inequality (9), Albert Ingham was able to provide the following, much simpler, proof [8]:

Proof.

we proceed by denoting a_n^p as b_n, and take the limit as $p \to \infty$. The right hand side becomes

$$\lim_{p\to\infty} \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{N} \alpha_n b_n = \lim_{p\to\infty} \left(1 + \frac{1}{p-1}\right)^p \sum_{n=1}^{N} \alpha_n b_n = e \sum_{n=1}^{N} \alpha_n b_n;$$

and on the left hand side, we will address each term of the sum,

$$\lim_{p\to\infty} \left(\frac{\alpha_1 b_1^{1/p} + \dots + \alpha_n b_n^{1/p}}{\alpha_1 + \dots + \alpha_n}\right)^p;$$

separately. For each such term, we can rewrite it as

$$\lim_{p\to\infty} \left(\sum_{i=1}^{n} w_i b_i^{1/p}\right)^p;$$

where $w_i = \alpha_i/\sigma_n$ and $\sum_{i=1}^{n} w_i = 1$. Letting $k = 1/p$, we can proceed as follows, making use of the fact that exp is a continuous function on $\mathbb{R}$:

$$\lim_{p\to\infty} \left(\sum_{i=1}^{n} w_i b_i^{1/p}\right)^p = \lim_{k\to 0} \sum_{i=1}^{n} w_i b^k$$

Returning to our original notation, this can be written as:

$$\lim_{p\to\infty}\left(\frac{\lambda_1 b_1^{1/p}+\cdots+\lambda_n b_n^{1/p}}{\lambda_1+\cdots+\lambda_n}\right)^p=\left(b_1^{\lambda_1}\cdots b_n^{\lambda_n}\right)^{1/\Lambda_n}.$$

Combined with our previous result when taking the limit of the right hand side of Hardy's inequality, we arrive at our desired result:

$$\sum_{n=1}^{\infty}\lambda_n\left(b_1^{\lambda_1}\cdots b_n^{\lambda_n}\right)^{1/\Lambda_n}\le e\sum_{n=1}^{\infty}\lambda_n b_n$$

and the constants $(\frac{p}{p-1})^p$ and p^p are the best such constants in each case respectively. Moreover, each of these inequalities may be deduced from the other.

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References

[1] Sheldon Axler. Measure, Integration & Real Analysis. Springer International Publishing, 2020.

[2] E. T. Copson. Note on series of positive terms. Journal of the London Mathematical Society, s1-2(1):9{12, 1927.

[3] G. H. Hardy. Remarks on three recent notes in the journal. Journal of the London Mathematical Society, s1-3(3):166{169, 1928.

[4] Godfrey Harold Hardy. Notes on some points in the integral calculus xli: On the convergence of certain integrals and series. Messenger of Mathematics, 44:163{166, 1915.

[5] Godfrey Harold Hardy. Notes on some points in the integral calculus li: On hilbert's double-series theorem, and some connected theorems concerning the convergence of in nite series and integrals. Messenger of Mathematics, 48:107{112, 1919.

[6] Godfrey Harold Hardy. Note on a 9 theorem of hilbert. Mathematische Zeitschrift, 6:314{317, 1920.

[7] Godfrey Harold Hardy. Notes on some points in the integral calculus lx: An inequality between integrals. Messenger of Mathematics, 54:150{156, 1925.

[8] Alois Kufner, Lech Maligranda, and Lars-Erik Persson. The prehistory of the hardy inequality. The American Mathematical Monthly, 113(8):715{732, 2006.