Sampling Young Tableaux and Contingency Tables

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1 Contingency Tables and Markov Chains

The problem motivating this work is that of sampling contingency tables. A contingency table A is a matrix with nonnegative integer entries whose rows and columns sum to some specified values. In other words, given vectors $r = (r_i)_{i=1}^m$ and $c = (c_j)_{j=1}^n$ of positive integers, a contingency table with row sums r and column sums c is some $A \in \mathrm{Mat}_{m \times n}(\mathbb{N})$ such that

$$r_i = \sum_{t=1}^{n} A_{i,t} \quad \text{and} \quad c_j = \sum_{t=1}^{m} A_{t,j} \tag{1}$$

for each $i \in [m]$ and $j \in [n]$ (we use the notation $[k] = \{1, \ldots, k\}$). Notice that it must be that $\sum r_i = \sum c_j$ for such a contingency table to possibly exist. We and call P the Markov chain's *transition matrix*. It can be easily seen that for any $t \in \mathbb{N}$,

$$\Pr(X_t = b \mid X_0 = a) = P^t(a, b). \qquad (4)$$

We define the *period* of state i to be $p_i := \gcd\{t \mid P^t(i, i) > 0\}$. We then say that a Markov chain is *aperiodic* if there exists a state i such that $p_i = 1$. We say a Markov chain is *irreducible* or *connected* if for every two states $i, j \in$, there exists some $t \in \mathbb{N}$ such that $P^t(i, j) > 0$. A Markov chain that is both aperiodic and irreducible is said to be *ergodic*.

For any probability distribution $_t$ over the elements of , we may think of $_{t+1} = {}_t P$ as the distribution acquired after performing a transition of the Markov chain on $_t$. We say that a distribution on is a *stationary distribution* if $= P$. With these definitions, we can state a fundamental result.

Theorem 1. *An ergodic Markov chain has a unique stationary distribution ; moreover, the Markov chain tends to in the sense that $P^t(a, b) \to {}_b$ as $t \to \infty$, for all $a, b \in$.*

This result implies that an ergodic Markov chain can be used to sample elements from a distribution close to over . We can start with any element $a \in$ and transition to other elements according to the rules defined by the Markov chain. How close the ending distribution is to is dependent on t, the number of transitions and the transition matrix. In general, the more transitions we perform, the closer the distribution gets to . Thus, a Markov chain is useful for sampling if its stationary distribution matches the desired distribution over we wish to sample from and if it can quickly converge to the stationary distribution.

One useful result in computing the stationary distribution of a Markov chain is the following:

Theorem 2. *Suppose P is the transition matrix of a Markov chain. If the function $'$: $\to [0, 1]$ satisfies*

$$' \quad)$$

The Markov chain then transitions from A to A' if the entries in A' are non-negative. Although it is easy to describe, researchers have had a difficult time analyzing the mixing time of this Markov chain.

In a paper from 2017, Kayibi et al. [12] attempt to show that this Markov chain mixes fast by using a canonical path argument. However, we managed to find a counterexample to one of the results used in the argument. Specifically, the paper states the following claim:

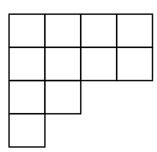
Proposition 3 (Corollary 8 from [12]). *Let N be the number of all $m \times n$ contingency tables of fixed row and column sums. The number of contingency tables having k fixed cells (in lexicographic ordering) is at most $N^{\frac{mn-k}{mn}}$.*

A counterexample to this proposition can be seen as follows: fix $n \in \mathbb{Z}^+$ and let $r = c = (1, 1, \ldots, 1)$. Then the set of $n \times n$ contingency tables with these row and column sums is exactly the set of tables acquired by permuting the rows of the $n \times n$ identity matrix. So in this case, $N = n!$. The set of these contingency tables with the first cell fixed to be 1 is the set of tables acquired by permuting the last n

2 Young Tableaux

We now turn our attention to the study of Young tableaux. Young tableaux, as we illustrate below, are innately connected to contingency tables, so studying how to sample these objects may provide us with a method of sampling contingency tables.

A *Young diagram* (sometimes called a Ferrers diagram) of shape = ($_1$; $_2$; : : : ; $_k$) is an array of k left-justified rows of cells such that row i has length $_i$ and $_1$ $_2$ $_k$ 1. A *standard Young tableau* of shape is a Young diagram filled with the integers from [n] where $n = \sum_{i=1}^{r} {}_i$ such that the integers are strictly increasing both from left to right within each row and from top to bottom within each column (so each integer from [n] appears precisely once). A *semistandard Young tableau* is a generalization in which the integers from [n] are allowed to appear more than once, and the row condition is relaxed to require that integers are only weakly increasing from left to right. Such a tableau is said to have weight = ($_1$; : : : ; $_n$) if each integer i appears $_i$ times. A standard Young tableau could be considered a semistandard Young tableau with weight = (1; 1; : : : ; 1). The following from left to right are examples of a Young diagram, a standard Young tableau, and a semistandard Young tableau, each with shape = (4; 4; 2; 1):



1	2	5	10
3	7	8	11
4	9		
6			

1	1

$c = (c_1; \ldots; c_n$

a random Young tableau from by randomly selecting the location of the largest entry and recursively lling out the rest of the tableau.

A *corner* of a Young diagram is a cell at the end of both its row and its column. Let c be the number of corners of , and let r_t be the row on which the t^{th} corner lies. Let ${}^{t} = ({}^{t}_{j})^{k}_{j=1}$ be the shape derived from by removing the t^{th} corner, so

$$ {}^{t}_{j} = \begin{cases} {}_{j} & \text{if } j \neq r_t \\ {}_{j} \quad 1 & \text{if } j = r_t \end{cases} : \tag{14}$$

Additionally, let ${}^{0} = ({}^{0}_{i})^{n}_{i=1}$ be the weight derived from by removing one count of the entry a, so

$$ {}^{0}_{i} = \begin{cases} {}_{i} & \text{if } i \neq a \\ {}_{i} \quad 1 & \text{if } i = a \end{cases} : \tag{15}$$

Since a is the largest entry in , it must be located on a corner of any Young tableau with weight . Given the t^{th} corner of , we can describe the probability that a is located at that corner of a Young tableau uniformly i.d9t(

Plancherel measure (for example, see [11]) and then sampling a Young tableau with shape . However, this special case of sampling contingency tables, as discussed in Section 1.1, is not particularly interesting, as the set of contin-

In 1979, Greene, Nijenhuis, and Wilf [7] provided an alternative probabilistic proof of the hook length formula. Their goal in reproving the result was to establish a better combinatorial explanation of why the hooks appear in the formula, as the proof provided by Frame, Robinson, and Thrall allows for no intuitive explanation. A convenient product of their new proof is that it gives an alternative and more e cient method of sampling standard Young tableaux, described as follows.

Given a Young diagram of shape with size n, randomly select a cell (i, j) with uniform probability $1/n$. Then, randomly select a new cell (i', j') from $H(i, j) \setminus \{(i, j)\}$ with uniform probability $1/(h(i, j) - 1)$. Select another cell from H (i50

In the following discussion, it will be useful to prove the following lemma about swaps that can be performed on corners.

Lemma 5. Let T 2 . If 2 [n 1] is located at a corner of T, then T[; + 1] is a valid Young tableau.

Proof. Consider the two spots of T local to both and + 1 which, in general, look like:

Because is at a corner, the row and column conditions ensure that these two diagrams can only overlap at ${}_1 = {}_2$ or ${}_2 = {}_1$.

Now, if and

$X_0' = X_0[\alpha;\alpha+1][\alpha+1;\alpha+2]\cdots[n-1;n]$ which has n at the end of row r_n. Now X_0' and Y_0 match in the location of n.

Now, remove n from both X_0' and Y_0, giving us two smaller Young tablueaux of size $n-1$ with shape $\lambda' = (\lambda'_r)_{r=1}^k$ with

$$\lambda'_r = \begin{cases} \lambda_r & \text{if } r \neq r_n \\ \lambda_r - 1 & \text{if } r = r_n \end{cases}. \tag{19}$$

Call these tableaux X_1 and Y_1. Now we can use the same process detailed above to transition X_1 to a Young tableau X_1' that matches Y_1 in the location of $n-1$. After removing $n-1$ from both X_1' and Y_1, we get two Young tableaux, X_2 and Y_2, of size $n-2$. We can repeat this process $n-2$ more times until the tableau derived from X_0 matches Y_0. Each swap that we perform has a positive probability of occurring in the Markov chain, so we have $P^t(X_0;Y_0) > 0$, where t is the total number of swaps. □

It follows easily that our Markov chain is ergodic.

Proposition 7. *MC_{swap} is ergodic.*

Proof. First note that for any $X \in \Omega$, $P(X;X) \geq \Pr(i = j) = 1/n$. Thus, the periodicity of X is 1, so the Markov chain is aperiodic. With Proposition 6, this implies that the Markov chain is ergodic. □

Now, by Theorem 1, we can conclude that this Markov chain has some stationary distribution π. Furthermore, just as we desire, the stationary distribution is the uniform distribution as shown here:

Proposition 8. *The stationary distribution π of MC_{swap} is uniform on Ω.*

Proof. Take any two distinct tableaux $X;Y \in \Omega$ such that Y differs from X by a single swap, i.e. there exist distinct $\alpha;\beta \in [n]$ such that $X[\alpha;\beta] = Y$. Then see that $P(X;Y) = \Pr(\{i;j\} = \{\alpha;\beta\}) = 2/n^2$, and by symmetry $P(Y;X) = 2/n^2$. For all other pairs $X;Y$ that do not differ by a single swap (either $X = Y$, or $P(X;Y) = 0$), we also have $P(X;Y) = P(Y;X)$. Thus, P is symmetric.

Let π' be the uniform distribution on Ω. Then for any $x;y \in \Omega$, we get

$$\pi'(x)P(x;y) = \pi'(y)P(y;x). \tag{20}$$

By Theorem 2, we know that π' is a stationary distribution for our Markov chain. Since the chain is ergodic, we know by Theorem 1 that the stationary distribution is unique. Thus, $\pi = \pi'$ is the uniform distribution on Ω. □

So we now know that we can use MC_{swap} to sample uniformly from the set of all standard Young tableaux of some fixed shape. Now we would like to bound the mixing time of this Markov chain. There are two primary methods to bound the mixing time of a Markov chain, one which uses "couplings" and one which uses "canonical paths."

3.3 Coupling for MC_{swap}

A Markovian *coupling* for a given Markov chain with space Ω and transition matrix P is a Markov chain (X_t, Y_t) on $\Omega \times \Omega$ with the following transition probabilities:

$$\begin{aligned} \Pr(X_{t+1} = a' \mid X_t = a, Y_t = b) &= P(a, a'), \\ \Pr(Y_{t+1} = b' \mid X_t = a, Y_t = b) &= P(b, b'). \end{aligned} \tag{21}$$

The coupling's transition matrix is often denoted $\hat{P}$. Essentially, it is a pair of two Markov chains run in parallel such that each individual chain looks like the Markov chain defined by P but which can be dependent on each other. Using such couplings can often be used to bound the mixing time of a Markov chain by using the following result often called the "Path Coupling theorem", first found in [2]:

Theorem 9. *For some Markov chain on Ω with transition matrix P, fix a coupling (X_t, Y_t). Let $G = (\Omega, E)$ be a graph and $d : E \to \mathbb{R}$ be a function that induces distances on $\Omega \times \Omega$. If there exists some $\beta > 0$ such that for all $\{a, b\} \in E$,*

$$E[d(X_{t+1}, Y_{t+1}) \mid X_t = a, Y_t = b] \leq (1 - \beta) d(a, b), \tag{22}$$

then

$$\tau(\varepsilon) \leq \frac{1}{\beta} \log \left(\frac{d_{max}}{\varepsilon} \right), \tag{23}$$

where $d_{max} = \max\{d(a, b) \mid (a, b) \in \Omega^2\}$.

[2] adds the remark that we can also get a bound on the mixing time if we assume the premise but with $\beta = 0$, albeit a weaker one.

To investigate whether such a method could be used to bound the mixing time of MC_{swap}, we construct the graph $G = (\Omega, E)$ where $E = \{\{a, b\} \mid P(a, b) > 0\}$. The distance function we define on G is the natural one; the whole of d on Ω^2 is induced by letting $d(a, b) = 1$ for every $\{a, b\} \in E$ with $a \neq b$.

Unfortunately, however, we can show that with this definition of G and d, the Path Coupling theorem cannot be used to bound the mixing time of MC_{paw}. We do this by defining a linear program that minimizes, over all possible couplings, the expected distance between two states after one transition of the Markov chain. It is defined more completely as follows.

Fix some $(a, b) \in E$. For every $(a', b') \in \Omega^2$, let $x_{a',b'} = \hat{P}((a,b),(a',b'))$ be a variable to be determined by our linear program. Our transition matrix $\hat{P}$ must define a coupling, so we must satisfy the constraints in eq. (21). These constraints are equivalent to

$$\begin{aligned} \sum_{b' \in \Omega} \hat{P}((a,b),(a',b')) &= P(a,a') \quad \text{for all } a' \in \Omega; \\ \sum_{a' \in \Omega} \hat{P}((a,b),(a',b')) &= P(b,b') \quad \text{for all } b' \in \Omega. \end{aligned} \tag{24}$$

Translating this into the variables in our linear program, we get the following constraints:

$$\begin{aligned} \sum_{b' \in \Omega} x_{a',b'} &= P(a,a') \quad \text{for all } a' \in \Omega; \\ \sum_{a' \in \Omega} x_{a',b'} &= P(b,b') \quad \text{for all } b' \in \Omega. \end{aligned} \tag{25}$$

Because all possible outcomes are represented by the probabilities $x_{a',b'}$, we may consider including the constraint

$$\sum_{a' \in \Omega} \sum_{b' \in \Omega} x_{a',b'} = 1; \tag{26}$$

but this is taken care of by the constraints in eq. (25) and the fact that P is a transition matrix, as

$$\sum_{a' \in \Omega} \sum_{b' \in \Omega} x_{a',b'} = \sum_{a' \in \Omega} P(a,a') = 1. \tag{27}$$

The only other constraints we need are those that force our variables to represent probabilities:

$$0 \le x_{a',b'} \le 1 \quad \text{for all } (a',b') \in \Omega^2. \tag{28}$$

Note, however, that the constraints listed in eq. (25) automatically provide an upper bound on the variables, so we do not need to include the upper bound here; that is, we only need

$$x_{a',b'} \ge 0 \quad \text{for all } (a',b') \in \Omega^2. \tag{29}$$

Now, we want to know if there exists a coupling such that the expected value found in eq. (22) is strictly less than $d(a; b) = 1$. Thus, we wish to minimize the following objective function:

$$E[d(X_t; Y_t) \mid X_t = a; Y_t = b] = \sum_{(a'; b') \in} d(a'; b') x_{a'; b'}: \qquad (30)$$

$13=18$. Thus, the constraints from eq. (25) translate to the following:

$$x_{a;a} + x_{b;a} + x_{d;a} + x_{e;a} + x_{f;a} + x_{g;a} = \frac{1}{18}; \quad (34)$$

$$x_{a;b} + x_{b;b} + x_{d;b} + x_{e;b} + x_{f;b} + x_{g;b} = \frac{16}{18}; \quad (35)$$

$$x_{a;c} + x_{b;c} + x_{d;c} + x_{e;c} + x_{f;c} + x_{g;c} = \frac{1}{18}; \quad (36)$$

$$x_{a;a} + x_{a;b} + x_{a;c} = \frac{13}{18}; \quad (37)$$

$$x_{b;a} + x_{b;b} + x_{b;c} = \frac{1}{18}; \quad (38)$$

$$x_{d;a} + x_{d;b} + x_{d;c} = \frac{1}{18}; \quad (39)$$

$$x_{e;a} + x_{e;b} + x_{e;c} = \frac{1}{18}; \quad (40)$$

$$x_{f;a} + x_{f;b} + x_{f;c} = \frac{1}{18}; \quad (41)$$

$$x_{g;a} + x_{g;b} + x_{g;c} = \frac{1}{18}: \quad (42)$$

Additionally, we still have the constraints $x_{i;j} \quad 0$ for each variable in consideration. By analyzing the tableaux, we get

$$d(a;c) = d(d;b) = d(e;b) = d(f;b) = d(g;b) = 2; \quad (43)$$

$$d(d;c) = d(e;c) = d(g;c) = 3; \quad (44)$$

$d(i;i) = 0$, and $d(i;j) = 1$ for all other pairs $(i;j)$ in consideration. This gives us the following objective function from eq. (30):

$$\text{minimize} \quad Z = x_{a;b} + 2x_{a;c} + x_{b;a} + x_{b;c} + x_{d;a} + 2x_{d;b} + 3x_{d;c} + x_{e;a} + 2x_{e;b} + 3x_{e;c} + x_{f;a} + 2x_{f;b} + x_{f;c} + x_{g;a} + 2x_{g;b} + 3x_{g;c}: \quad (45)$$

Now that we have the linear program de ned, we want to show that its optimum value is strictly larger than $d(a;b) = 1$, as this implies that the Path Coupling theorem does not apply. We will do this by considering the dual of our linear program:

$$\text{maximize} \quad Z^0 = \frac{1}{18}y_1 + \frac{16}{18}y_2 + \frac{1}{18}y_3 + \frac{13}{18}y_4 + \frac{1}{18}y_5 + \frac{1}{18}y_6 + \frac{1}{18}y_7 + \frac{1}{18}y_8 + \frac{1}{18}y_9 \quad (46)$$

subject to

$y_1 + y_4$	0	$y_1 + y_7$	1	(47)
$y_2 + y_4$	1	$y_2 + y_7$	2	(48)
$y_3 + y_4$	2	$y_3 + y_7$	3	(49)
$y_1 + y_5$	1	$y_1 + y_8$	1	(50)
$y_2 + y_5$	0	$y_2 + y_8$	2	(51)
$y_3 + y_5$	1	$y_3 + y_8$	1	(52)
$y_1 + y_6$	1	$y_1 + y_9$	1	(53)
$y_2 + y_6$	2	$y_2 + y_9$	2	(54)
$y_3 + y_6$	3	$y_3 + y_9$	3	(55)

Now, consider the assignment $y($

(where $E = \{(u,v) \mid P(u,v) > 0\}$) from x to y labeled $\gamma_{x,y}$. Then we let the congestion of an edge be

$$\text{Congestion}(u,v) = \frac{1}{\pi(u)P(u,v)} \sum_{\substack{\gamma_{x,y} \\ (u,v)\in\gamma_{x,y}}} \pi(x)\pi(y)|\gamma_{x,y}|. \tag{56}$$

We then have the following result:

Theorem 11. Let $\rho = \max\{\text{Congestion}(u,v) \mid (u,v) \in \Omega^2\}$. Then

$$\tau(\epsilon) \le 2\rho\left(2\ln\frac{1}{\epsilon} + \ln\frac{1}{\min \pi(x)}\right). \tag{57}$$

Thus, if we can describe canonical paths such that we can find a polynomial bound on the maximum congestion of an edge, we can get a polynomial bound on the mixing time of our Markov chain.

For MC_{pw}, we define canonical paths using the process established in the proof of Proposition 6. Fix two tableaux $u, v \in \Omega$. Locate the position p of n in v. Starting with $w = v$, use swaps to increment the number at p in w by 1 repeatedly until n is also located at p for w. Repeat this process for $n-1$, $n-2$, etc., until w is identical to v. As justified in the proof of Proposition 6, each of the intermediate tableaux are valid Young tableaux, so this process defines a canonical path $\gamma_{u,v}$ from vertex u to vertex v in our graph.

With these paths established, we need to bound the congestion of the edges of our graph. For a given pair of Young tableaux $x, v \in \Omega$, we have the following:

$$\text{Congestion}(u,v) = \frac{1}{\pi(u)P(u,v)} \sum_{\substack{\gamma_{x,y} \\ (u,v)\in\gamma_{x,y}}} \pi(x)\pi(y)|\gamma_{x,y}| \tag{58}$$

$$\le \frac{1}{|\Omega|} \sum_{\substack{\gamma_{x,y} \\ (u,v)\in\gamma_{x,y}}} |\gamma_{x,y}| \tag{59}$$

$$\le \frac{n(n+1)}{2|\Omega|} \sum_{\substack{\gamma_{x,y} \\ (u,v)\in\gamma_{x,y}}} 1. \tag{60}$$

Unfortunately, however, we do not currently have a bound on the size of the set $\{(x,y) \in \Omega^2 \mid (u,v) \in \gamma_{x,y}\}$ that yields a polynomial bound on the congestion of the edge (u,v). We leave this for future work.

4 Sampling Semistandard Young Tableaux

We now switch to the problem of sampling semistandard Young tableaux. Recall that the RSK correspondence maps contingency tables to pairs of semistandard Young tableaux of the same shape. Thus, this more generalized case is more interesting to us than the case of sampling standard tableaux. Furthermore, once we fix the row and column sums for our contingency tables, the weights of the corresponding Young tableaux are determined while their shapes are not. Thus, our overall goal is to sample pairs of semistandard Young tableaux with fixed weights and the same, but unfixed, shape.

4.1 Complexity of counting Young tableaux

Because of the strong connection between Kostka numbers (and by extension, Young tableaux) and other areas of mathematics, a lot of work has gone into understanding and calculating these coefficients (for example, see [18]). However, definitive results have remained elusive, possibly due to the complexity of the problem. In fact, in 2006, Hariharan Narayanan [17] showed that the problem of computing arbitrary Kostka numbers is #P-complete, meaning that unless $P = NP$, there does not exist an algorithm that can compute these numbers in polynomial time. However, even if $P \neq NP$, the question of example, consider the following Young tableaux:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline 4 & 5 \\ \hline\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 3 & 5 \\ \hline\end{array} : \tag{61}$$

These two tableaux both have the same shape and weight, so we would need to nd a way of getting from one to the other using swaps. Recall that the constraints of semistandard Young tableaux are that the rows need to be weakly increasing while the columns need to be strictly increasing. Therefore, there are no swaps we can perform on either of these tableaux that yield a valid Young tableau. Consequently, there is no way of getting from one tableau to the other using only swaps, and our Markov chain is not connected. Thus, we must consider a di erent type of Markov chain in the semistandard case.

4.3 A Markov chain for variable shapes

We now propose a Markov chain to sample from all Young tableaux of a given

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row, once b is moved to the second row, the entries located above b throughout the process will never be any larger than a and, hence, will satisfy the column constraints of a Young tableau. Since all row and column constraints are satisfied, each intermediate tableau will be a valid Young tableau. In this way, considered but not fully explored is that which calculates the total pairwise difference between two tableaux' entries. To define this more rigorously, let X and Y be two standard Young tableaux with the same shape. Let the entries of X in lexicographic order be $(a_1; a_2; \dots; a_n)$ and those of Y be $(b_1; b_2; \dots; b_n)$.

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n = Total [shape];
nv = Length [yts];
vertices = Range[nv];
edges = Select[Subsets
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