1 Introductory background

1.1 Discrete Hausdor dimension for families of nite sets

In the literature[9], one of the equivalent ways of de ning the Hausdor dimension of a compact set relies on the computation of the following quantity: the energy integral.

De nition 1. Given 0 d, the

(compactness is not even a strictly necessary condition) and relying on measures (which are easy to manipulate), the Hausdorff dimension has a central role in fractal geometry indicating, roughly speaking, how much space a set occupies near to each of its points . Its most fundamental definition (which is equivalent to Definition 2[3] and also explains why it can capture fractal properties of a set and the way it distributes in space) is the following:

Definition 5. Given a set E, its Hausdorff dimension is

$$\inf\{s \geq 0 : H^s(E) = 0\} = \sup\{s \geq 0 : H^s(E) = \infty\} \tag{7}$$

where $H^s(E)$ is the Hausdorff measure of E and it is defined as

$$\lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : E \subseteq \bigcup_{i=1}^{\infty} U_i ; |U_i| < \delta \ \forall i \in \mathbb{Z} \right\} \tag{8}$$

The Hausdorff dimension also quantifies the roughness or smoothness of a time series in the limit as the observational scale becomes infinitesimally fine[4]. With respect to this, referring to the previous example, fractal analysis can be applied meaningfully not only on the time axis of the considered time series (mentioned before), but also on the values of the sales. Such analyisis, in fact, would reflect interestingly the volatility of the recorded values. For example, it has been shown that the fractional Brownian path (used to model stock market prices) has Hausdorff dimension > 1, reflecting the volatility of the data. After noticing this, then, we arrive at a very interesting situation where we have a set of effective dimension < 1 (as it is embedded in a uni-dimensional space) on the time axis, combined with the volatile data modeled by a function whose graph has dimension > 1. Thus, exploring discrete families of point sets resembling finer and finer time series finds its motivation. In this paper, we will focus on the volatile part of the time series and, for this reason, the set of points on the time axis is taken to be equidistributed as in (4) to simplify things. Finally, by looking at Definition 5, it is noticeable that the Haussdorf dimension of any discrete set is 0. Since the time series are discrete objects, then, it becomes crucial to restate the definition of a discrete Hausdorff dimension as done in Definition 4 in order to obtain meaningful results.

1.3 Statement of the main theorem and preliminary results

The main theorem proved in this paper is the following:

Theorem 6. Given $f : [0,1]^{d-1} \to [0,1]$, let $P = \{P_n\}$ be the time series with

$$P_n = \left\{ (j/q; f(j/q)) : j \in \mathbb{Z}^{d-1} \cap [0,q)^{d-1} \right\}; \qquad n = q^{d-1} \tag{9}$$

For any $s \in [d-1, d)$, there exists f such that $\dim_H(P) = s$.

Before proving the just stated theorem, we mention several results proved in the paper written during the TRIPODS NSF REU-GradForAll 2021[1] (in preparation).

Firs4l41 0 /F70 8.9664 7rs4l41 TJ -420.tain J -420.4l(g)]Tme< re7 Td23 eleon .0i where here j and j' range over $\mathbb{Z}^{d-1} \cap [0,q)^{d-1}$. By a change of variables $j - j'$ to j, the above become:

$$
\begin{aligned}
& q^{-2(d-1)+s} \sum_{j' \in \mathbb{Z}^{d-1} \cap [0,q)^{d-1}} \; \sum_{\substack{j + j' \in \mathbb{Z}^{d-1} \cap [0,q)^{d-1} \\ j \neq 0}} |j|^{-s} \\
&= q^{-2(d-1)+s} \sum_{j' \in \mathbb{Z}^{d-1} \cap [0,q)^{d-1}} \; \sum_{\substack{j \in \mathbb{Z}^{d-1} \cap \prod_{i=1}^{d-1} [-j'_i, q-j'_i) \\ j \neq 0}} |j|^{-s} \\
&\le q^{-2(d-1)+s} q^{d-1} \sum_{\substack{j \in \mathbb{Z}^{d-1} \cap (-q,q)^{d-1} \\ j \neq 0}} |j|^{-s} \le C q^{-(d-1)+s} \int_0^q r^{-s+d-2} \, dr \qquad (12) \\
&= C q^{-(d-1)+s} q^{d-1-s} = C
\end{aligned}
$$

by the integral test, where here C is a constant depending on d and s only. Note that the integral converged for values $s < d-1$. The conclusion of Lemma 7 follows by (6). □

Next, we present a Lemma that gives us a lower 000

Proof. The goal is to show that $I_s(P_n)$ is unbounded for $s >$, and provide the quantitative lower bound (16) provided P_n is confined to E, and where the constant implicit in the notation depends on s and but is independent of n. We write

$$
\begin{aligned}
|p - p'|^{-s} &= s \int_{|p-p'|}^{\infty} r^{-s-1}\, dr \\
&= \int_0^{\infty} 1_{[|p-p'|;\infty)}(r)\, r^{-s-1}\, dr \\
&= s \int_0^{\infty} 1_{[0;\infty)}(r - |p - p'|)\, r^{-s-1}\, dr;
\end{aligned}
$$

1.4 Review of relevant theorems from probability theory

In this section, we review some standard theorems from the literature of probability theory [2] that are later used in the paper.

Recall that, any real valued random variable X defined on a probability space $(\Omega; \mathcal{F}; P)$ induces a probability push-forward measure μ on the measurable space $(\mathbb{R}; \mathcal{B}(\mathbb{R}))$ by setting $\mu(A) = P(X^{-1}(A)) = P(X \in A)$ for any $A \in \mathcal{B}(\mathbb{R})$. Moreover, we define the cumulative distribution function of X to be $F(x) = P(X \leq x) = \mu(X \in (-\infty; x])$ and, if this one can be written as $F(x) = \int_{-\infty}^{x} f(y)dy$, we say that X has density function f. The first theorem presents

Proof. Let $h(x; y) = 1_{\{x+y \le z\}}$. Then, by Theorem 11:

$$P(X + Y \le z) = E\left[1_{\{X+Y \le z\}}\right] = \int_R \int_R 1_{\{x+y \le z\}} (dx)\ (dy)$$
$$= \int_R \int_R 1_{\{x \le z - y\}} (dx)\ (dy) = \int_R E\left[1_{\{X \le z\}}\right.$$

Proposition 15. For any $x; y$ and any θ such that $|x - y| < 1/2b^2$:

$$E_H\left[|(x; f_\theta(x)) - (y; f_\theta(y))|^{-s}\right] \leq C|x - y|^{(1-\alpha-s)} \tag{37}$$

for any $s \in (1; 2-\alpha)$

Proof. Fix $x; y$ such that $0 < |x - y| \leq \frac{1}{2b^2}$. Now, let $z = f_\theta(x) - f_\theta(y)$. Note that, for fixed values of x and y, $z : H \to \mathbb{R}$ is a real-valued random variable (i.e. measurable function). Let $h(z)$ be the density function of the just mentioned random variable. Then, performing two changes of variable (one to rewrite the expectation in terms of the density and the other one consisting of the simple substitution $z = |x - y|\omega$):

$$\begin{aligned} E_H\left[|(x; f_\theta(x)) - (y; f_\theta(y))|^{-s}\right] &= \int_{-\infty}^{\infty} \frac{h(z)}{((x-y)^2 + z^2)^{\frac{s}{2}}}\,dz \\ &= \int_{-\infty}^{\infty} \frac{h(|x-y|\omega)\,|x-y|}{|x-y|^s (1+\omega^2)^{\frac{s}{2}}}\,d\omega \\ &\leq \sup_z h(z)\,|x-y|^{1-s} \int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^{\frac{s}{2}}}\,d\omega \\ &\lesssim \sup_z h(z)\,|x-y|^{1-s} \end{aligned} \tag{38}$$

The last step followed since the above integral is convergent for $s > 1$ and we are taking $s \in (1; 2-\alpha)$. After obtaining this bound, to complete the proof, it is then sufficient to show that $h(z) \leq C|x - y|^{-\alpha}$ for any z.
In order to prove such result, first, note that the random variable z can be rewritten as:

$$\begin{aligned} z &= f_\theta(x) - f_\theta(y) \\ &= \sum_{n=0}^{\infty} b^{-n\alpha}\left(\cos(2\pi(b^n x + \theta_n)) - \cos(2\pi(b^n y + \theta_n))\right) \\ &= \sum_{n=0}^{\infty} 2b^{-n\alpha} \sin\left(\frac{2\pi b^n(y-x)}{2}\right) \sin\left(2\pi\left(\frac{b^n(y+x)}{2} + \theta_n\right)\right) \\ &= \sum_{n=0}^{\infty} q_n \sin(r_n + 2\pi\theta_n) \\ &= \sum_{n=0}^{\infty} z_n \end{aligned} \tag{39}$$

where $q_n = 2b^{-n\alpha}\sin(\pi b^n(y-x))$ and $r_n = 2b^{-n\alpha}\sin(\pi b^n(y+x))$ are independent of any θ_n (i.e. independent of the infinite sequence θ). Next, for any n, the cumulative distribution of the random variables z_n is:

$$\begin{aligned} P(q_n \sin(r_n + 2\pi\theta_n) \leq y) &= \frac{1}{2\pi}\left(1 - |\{x \in [r_n; r_n + 2\pi] : q_n \sin(r_n + 2\pi\theta_n) > y\}|\right) \\ &= \frac{1}{2\pi}\left(\pi - 2\arccos\left(\frac{y}{q_n}\right)\right) \end{aligned} \tag{40}$$

for $y \in [-q_n; q_n]$. This is because $r_n + 2\pi\theta_n$ in the argument of the sin function is uniformly distributed on an interval of length 2π and also since

$$\begin{aligned} y &= q_n \sin\left(\frac{\pi}{2} - \frac{|\{x \in [r_n; r_n + 2\pi] : q_n \sin(r_n + 2\pi\theta_n) > y\}|}{2}\right) \\ &= q_n \cos\left(\frac{|\{x \in [r_n; r_n + 2\pi] : q_n \sin(r_n + 2\pi\theta_n) > y\}|}{2}\right) \end{aligned} \tag{41}$$

Since the z_n are continuous random variables, we can then obtain their probability density distributions:

$$\begin{aligned} h_n(z_n) &= \frac{d}{dz_n} P(q_n \sin(r_n + 2\pi\theta_n) \le z_n) \\ &= \frac{d}{dz_n}\frac{1}{2}\left(1 - 2\arccos\frac{y}{q_n}\right) \\ &= \begin{cases} \frac{1}{\pi\sqrt{q_n^2 - z_n^2}} & z_n \in [-q_n, q_n] \\ 0 & z_n \notin [-q_n, q_n]. \end{cases} \end{aligned} \tag{42}$$

Then, using the result of theorem (13), since $z = \sum_{n=0}^{\infty} z_n$, we have that its density is the infinite convolution $h = h_0 * h_1 * \cdots$. Furthermore, noting that the maximum value of a probability density cannot increase under convolution with another probability density, in order to bound $h(z)$, it is sufficient to find a bound on a finite convolution $h_j * \cdots * h_k$ for some values of j and k. Recall that, by hypothesis, $0 < |x - y| < \frac{1}{2b^2}$. Choose an integer $k \ge 2$ such that $\frac{1}{2}b^{-k-1} < |x - y| < \frac{1}{2}b^{-k}$. Thus:

$$\frac{\pi}{2b^3} < \pi 2 b^{k-2}\frac{y - x}{2} < \pi 2 b^{k}\frac{y - x}{2} \le \frac{\pi}{2}; \tag{43}$$

and hence

$$|q_n| > 2\sin\left(\frac{\pi}{2b^3}\right) b^{-k} > 2\sin\left(\frac{\pi}{2b^3}\right) b \tag{44}$$

for $n = k-2, k-1, k$. Now, notice that with a change of variable z_n to $q_n z_n$:

$$\begin{aligned} \|h_n\|_p^s &= 2|q_n|^{1-s}\int_0^1 \frac{1}{\pi\sqrt{1 - z_n^2}^{\,p}}\,dz_n \\ &\le 2|q_n|^{1-s}\int_0^1 \frac{1}{\sqrt{1 - z_n^2}^{\,p}}\,dz_n \\ &= 2|q_n|^{1-s}\int_0^1 \frac{1}{|z_n|^{\frac{p}{2}}}\,dz_n \end{aligned} \tag{45}$$

where the above integral converges for values $p < 2$. Then, combining (44) and (45), for $n = k-2, k-1, k$, we have that:

$$\|h_n\|_{\frac{3}{2}} = K|q_n|^{-\frac{1}{2}} \le K'|x - y|^{-\frac{1}{3}} \tag{46}$$

where K is an absolute constant and K' depends only on b. Then, by an application of Young's inequality,

$$\|h_{k-1} * h_k\|_3 \le \|h_{k-1}\|_{\frac{3}{2}}\|h_k\|_{\frac{3}{2}} \tag{47}$$

and combining Hölder's inequality with (47) and (46), we obtain:

$$\begin{aligned} h(z) &= h_0 * h_1 * \cdots * h_{k-2} * h_{k-1} * h_k \\ &\le \|h_{k-2}\|_{\frac{3}{2}}\|h_{k-1} * h_k\|_3 \le \|h_{k-2}\|_{\frac{3}{2}}\|h_{k-1}\|_{\frac{3}{2}}\|h_k\|_{\frac{3}{2}} \\ &\le K''|x - y|^{-1} \end{aligned} \tag{48}$$

for any z, completing the proof. □

$$II = q^{2} \sum_{\substack{j \neq j' \\ j, j' \in [0; q) \cap Z \\ |j - j'| > \frac{q}{2b^2}}} E_H \left(\frac{j}{q}; f \quad \frac{j}{q} \qquad \frac{j'}{q}; f \quad \frac{j'}{q} \right)_{s}^{!} \tag{57}$$

The second term is easy to bound as:

$$II$$

where

$$I = q^{-2(d-1)} \sum_{\substack{j_1 \neq j_1' \\ j_1, j_1' \in [0,q) \cap \mathbb{Z}}} E_H \left(\sum_{j, j' \in [0,q)^{d-2} \cap \mathbb{Z}^{d-2}} \left(\left| \frac{j_1}{q} - \frac{j_1'}{q} \right|^2 + \left| f\left(\frac{j_1}{q}\right) - f\left(\frac{j_1'}{q}\right) \right|^2 + \left| \frac{j}{q} - \frac{j'}{q} \right|^2 \right)^{-\frac{s}{2}} \right) \tag{62}$$

$$II = q^{-2(d-1)} \sum_{\substack{j_1 = j_1' \\ j_1, j_1' \in [0,q) \cap \mathbb{Z}}} E_H \left(\sum_{\substack{j \neq j' \\ j, j' \in [0,q)^{d-2} \cap \mathbb{Z}^{d-2}}} \left(\left| \frac{j_1}{q} - \frac{j_1'}{q} \right|^2 + \left| f\left(\frac{j_1}{q}\right) - f\left(\frac{j_1'}{q}\right) \right|^2 + \left| \frac{j}{q} - \frac{j'}{q} \right|^2 \right)^{-\frac{s}{2}} \right) \tag{63}$$

where $j = (j_2, \ldots, j_{d-1})$. We just need to worry about bounding term I since it is greater than term II. To see this, note that any addend in the inner sum of II is comparable to an addend of the inner sum of II but, at the same time, the outer sum of I contains way more terms than the outer sum of II (due to the difference between the conditions $j_1 = j_1'$ and $j_1 \neq j_1'$). To bound the first term, we can further decompose it in two pieces $I = III + IV$ where:

$$III = q^{-2(d-1)} \sum_{\substack{j_1 \neq j_1' \\ j_1, j_1' \in [0,q) \cap \mathbb{Z} \\ |j_1 - j_1'| \leq \frac{q}{2b^2}}} E_H \left(\sum_{j, j' \in [0,q)^{d-2} \cap \mathbb{Z}^{d-2}} \left(\left| \frac{j_1}{q} - \frac{j_1'}{q} \right|^2 + \left| f\left(\frac{j_1}{q}\right) - f\left(\frac{j_1'}{q}\right) \right|^2 + \left| \frac{j}{q} - \frac{j'}{q} \right|^2 \right)^{-\frac{s}{2}} \right) \tag{64}$$

$$IV = q^{-2(d-1)} \sum_{\substack{j_1 \neq j_1' \\ j_1, j_1' \in [0,q) \cap \mathbb{Z} \\ |j_1 - j_1'| > \frac{q}{2b^2}}} E_H \left(\sum_{j, j' \in [0,q)^{d-2} \cap \mathbb{Z}^{d-2}} \left(\left| \frac{j_1}{q} - \frac{j_1'}{q} \right|^2 + \left| f\left(\frac{j_1}{q}\right) - f\left(\frac{j_1'}{q}\right) \right|^2 + \left| \frac{j}{q} - \frac{j'}{q} \right|^2 \right)^{-\frac{s}{2}} \right) \tag{65}$$

The last term is bounded as follows:

$$\begin{aligned} IV &\leq q^{-2(d-1)} \sum_{\substack{j_1 \neq j_1' \\ j_1, j_1' \in [0,q) \cap \mathbb{Z} \\ |j_1 - j_1'| > \frac{q}{2b^2}}} \sum_{j, j' \in [0,q)^{d-2} \cap \mathbb{Z}^{d-2}} \left| \frac{j_1}{q} - \frac{j_1'}{q} \right|^{-s} \\ &\leq q^{-2(d-1)} q^2 q^{2(d-2)} (2b^2)^s = (2b^2)^s \end{aligned} \tag{66}$$

In the inner sum of III, the quantity $\left| \frac{j_1}{q} - \frac{j_1'}{q} \right|^2 + \left| f\left(\frac{j_1}{q}\right) - f\left(\frac{j_1'}{q}\right) \right|^2$ is constant (i.e. does not dep71(.072r -3of)]nJ 0 -1782n6w587

$$\sum_{j;\, j' \in [0;q)^{d-2} \cap \mathbb{Z}^{d-2}} \quad {}^{2} + \frac{j}{q} \quad \frac{j'}{q}$$

) the second one by integral test. Additionally since we considerF37 5.9564 Tf 9.21627