HINGES IN Z_p^d AND APPLICATIONS TO PINNED DISTANCE SETS

BRIAN MCDONALD

CONTENTS

1. INTRODUCTION

1.1. Background. In this paper, we investigate the number of chains in a subset of Z_p^d, the d-dimensional vector space over the finite field with p elements, for a prime p.

Definition 1.1. *A* ***chain*** *in E Z_p^d of length n with common distance t is a sequence $x_1, x_2, ..., x_{n+1}$ where for $i = 1, ..., n$, $||x_{i+1} - x_i|| = t$.*

Here, and throughout the paper, $||x|| = x_1^2 + x_2^2 + \cdots + x_d^2$. Note that chains in E are equivalent to paths on the distance graph of E. Theorem 1.1 of [1] gives an estimate for the number of paths of a certain length, which is a very similar result to Theorem 1.4 in this paper; in fact, it is essentially the same but with a better constant on the error term. We give a different proof, and also extend the idea to dot products as well as distances. In other words, we consider sequences with the property $||x_{i+1} \cdot x_i|| = t$.

After obtaining our result estimating the number of chains, we consider an application of the case $n = 2$ to estimating the size of pinned distance sets.

Definition 1.2. *The distance set of E pinned at x E is*

$$\Delta_x(E) = \{||x - y|| : y \quad E\} \tag{1}$$

In fact, the idea we employ here could be generalized further to relate chains of length $2n$ to pinned chains of length n.

Date: May 9, 2016.

1.2. **Main Results.**

Definition 1.3. *Let $\Lambda_t^n(E)$ be the number of chains in E of length n, with common distance t, and let $\Gamma_t^n(E)$ be the number of dot-product chains in E of length n with*

Proof.

so the $m = m$ terms actually give a negative contribution, so we may just ignore them. Therefore, we have the desired bound on the error term.

□

Corollary 2.5.

$$\Lambda_t^n(E) = \sum_x E(x)\psi_n(x) = ||E\psi_n||_1, \quad (22)$$

$$\Gamma_t^n(E) = \sum_x E(x)\phi_n(x) = ||E\phi_n||_1 \quad (23)$$

Proof. This is the special case $k = n$. □

We have now reduced the original problem to estimating $||E\psi_n||_1$ and $||E\phi_n||_1$, which we will do using a few recursive inequalities. Until this stage, we have been simultaneously handling the distance case and the dot product case in paralell, since they have been structurally identical. Now the two proofs will diverge - not because they are different in a particularly meaningful way, but because the small differences between the two cases will start to add up, and so it would be less efficient to do them

Proof. This is the same idea as the previous lemma, but with $f(x) = E(x)\psi_k(x)$, $g(y) = E(y)\psi_{k-1}$. We have

$$\begin{aligned} b_k^2 &= \sum_x E(x)\psi_k(x)^2 = \sum_{x,y} E(x)E(y)\psi_k(x)\psi_{k-1}(y)S_t(x-y) \\ &\leq p^{-1}||E\psi_k||_1||E\psi_{k-1}||_1 + 2p^{\frac{d-1}{2}}||E\psi_k||_2||E\psi_{k-1}||_2 \\ &= p^{-1}a_k a_{k-1} + 2p^{\frac{d-1}{2}} b_k b_{k-1} \end{aligned} \tag{27}$$

□

Now the idea is to combine these two inequalities to bound a_k, b_k above by induction. Then we will use that upper bound to prove our main result.

Lemma 3.3. *For some constant C_k depending only on k, we have*

$$a_k \leq C_k \frac{|E|^{k+1}}{p^k}, \quad \textit{and} \quad b_k \leq C_k \frac{|E|^{k+\frac{1}{2}}}{p^k} \tag{28}$$

Proof. It is clearly true in the case $k = 0$ with the choice of constant $C_0 = 1$. Assume it is true for indices less than k. We have by Lemma 3.1 that

$$\begin{aligned} \left(p^{-1}a_k a_{k-1}\right) &\leq p^{-2}|E|a_{k-1}^2 + 2p^{\frac{d-3}{2}}|E|^{\frac{1}{2}}a_{k-1}b_{k-1} \\ &\leq C_{k-1}^2\frac{|E|^{2k+1}}{p^{2k}} + 2C_{k-1}^2\frac{|E|^{2k+1}}{p^{2k}} \\ &= 3C_{k-1}^2\frac{|E|^{2k+1}}{p^{2k}} \end{aligned} \tag{29}$$

Also,

$$2p^{\frac{d-1}{2}}b_{k-1} \leq 2C_{k-1}\frac{|E|^{k+\frac{1}{2}}}{p^k} \tag{30}$$

The following simple algebra is helpful: for $A, B > 0$, if $x \leq \sqrt{A + Bx}$, then we can solve the corresponding quadratic polynomial to find that

$$x \leq \frac{B + \sqrt{B^2 + 4A}}{2} \leq \max(B, \sqrt{B^2 + 4A}) \leq \sqrt{B^2 + 4A} \tag{31}$$

We use this inequality with

$$\begin{aligned} x &= b_k, \\ A &= 3C_{k-1}^2\frac{|E|^{2k+1}}{p^{2k}}, \\ B &= 2C_{k-1}\frac{|E|^{k+\frac{1}{2}}}{p^k} \end{aligned} \tag{32}$$

And we obtain

$$b_k \le \left(4C_{k-1}\frac{|E|^{2k+1}}{p^{2k}} + 12C_{k-1}^2\frac{|E|^{2k+1}}{p^{2k}} \right)^{\frac{1}{2}}$$
$$\le 4C_{k-1}\frac{|E|^{k+\frac{1}{2}}}{p^k} \qquad (33)$$

Therefore, $C_k = 4^k$

Where

$$
\begin{aligned}
R &:= \sum_{i=0}^{k-1} \frac{|E|^i}{p^i} R_{k-i} \\
&\leq 2p^{\frac{d-1}{2}} \frac{|E|^k}{p^{k-1}} \sum_{i=0}^{k-1} 4^{k-i} \\
&= 2p^{\frac{d-1}{2}} \frac{|E|^k}{p^{k-1}} \left(\frac{4^{k+1} - 4}{3} \right) \qquad (37)
\end{aligned}
$$

In light of the fact that $a_k = \Lambda_t^k(E)$, we are done. □

3.2. **Dot Products.**

Similarly to before, we will have $d_k \leq \sqrt{}$

4. APPLICATION TO PINNED DISTANCE SETS

Definition 4.1 (Pinned Distance Sets)**.** *For* $x \in E \subseteq \mathbb{Z}_p^d$, *let*

$$\Delta_x(E) = \{||x - y|| : y \in E\} \tag{48}$$

We call this the distance set of E *pinned at* x*.*

Our goal is to find a lower bound for $|\Delta_x(E)|$ via estimating certain sums over $\mathbb{Z}_p^d$. Consider the quantity

$$\nu_x(t) := |\{y \in E : ||x - y|| = t\}| = \sum_y E(y)S_t(x - y) \tag{49}$$

Note that for every $y \in E$, there is precisely one t (namely $t = ||x - y||$) for which $y \in \{y \in E : ||x - y|| = t\}$. This trivial observation, along with an application of Cauchy-Schwarz, allows us to write

$$|E|^2 = \left(\sum_t \nu_x(t)\right)^2 \leq |\Delta_x(E)| \sum_t \nu_x(t)^2 \tag{50}$$

Therefore, bounding $\Delta_x(t)$ below can be done by bounding $\sum_t \nu_x(t)^2$ above. Rather than doing this for a fixed x, we will look at what happens when we sum over x.

$$\begin{aligned}\sum_{x,t} \nu_x(t)^2 &= \sum_t \sum_{x,y,z} E(x)E(y)E(z)S_t(x - y)S_t(x - z)\\ &\leq \sum_t \Lambda_t^2(E) \leq p\left(\frac{|E|^3}{p^2} + 40p^{\frac{d-1}{2}}\frac{|E|^2}{p}\right)\\ &= \frac{|E|^3}{p} + 40\end{aligned}$$

5. ACKNOWLEDGEMENTS

I would like to thank Alex Iosevich for all of his support, academic and otherwise,