Model Theory

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1 Introduction

The goal of this paper is to provide a reasonably complete exposition of the Ax-Kochen theorem and its (partial) resolution of Artin's conjecture, in addition to a small selection of other applications of model theory and mathematical logic to algebra. This is based largely on Chang and Keisler's

On the other hand, the connections between syntax and semantics are important too. The language

Finally, it remains to show that M is truly a model of T

Theorem 2.7

4 Ultraproducts

In proving the completeness theorem above, we constructed a model out of constants, and saw that the construction itself had interesting applications (though many of a negative nature). Another important kind of construction in model theory is that of the ultraproduct. Essentially, the ultraproduct will allow us to construct the "average" model from a family of models. This construction can give an alternative proof of the compactness theorem, though the control it gives over the cardinality of the resulting model is not fine enough to prove some of the theorems stated above.

Definition 4.1. Given a set X, a subset F of $P(X)$ is called a **filter** on X if:

1. F is nonempty.

2. F

6 Ax-Kochen

Our final goal is to prove the Ax-Kochen theorem, closely following the presentation in [1].

Definition 6.1. A valued field F, with cross-section, is a model of the two-sorted language

$$L = \{F, +$$

Proof. First, we may assume that F and G are saturated fields of cardinality ω_1 - the set of all first-order sentences true of F is complete and consistent, so it has a saturated model of cardinality $||L||^+ = |2^\omega| = \omega_1$ (contingent on the GCH), and likewise for G. Clearly, if F and G are saturated then their value groups and residue-fields are saturated. Moreover, outside the case where $val(F) = val(G) = \{1\}$, the value groups and residue fields all have cardinality ω_1 (in that trivial case, $F = F \quad \equiv G \quad = G$ and there is nothing to prove).

We will write $f_1 : F_1 \leftrightarrow G_1$ iff f_1 is an isomorphism and

$$(val(F), x)_{x \quad val(F_1)} \equiv (val(G), x)_{x \quad val(G_1)}$$

Our goal is to show that F and G are isomorphic using a back-and-forth argument like that used to show that any two elementarily equivalent models of the same cardinality are isomorphic. An outline of the induction is as follows:

1. Since $F \quad \equiv G$ and $|F \quad| = |G \quad| = \omega_1$, they are isomorphic. It follows from Hensel's lemma that they are algebraically closed in F and G, respectively. This is our base case.

2. Suppose F_1 and G_1 are algebraically closed valued subfields of F and G, containing their respective residue fields, and where $val(F_1) = val(G_1)$ is countable. Let $f_1 : F_1 \leftrightarrow G_1$ extending $f_0 : F \quad \leftrightarrow G \quad$. For every $x \quad F - F_1$ there exist algebraically closed valued subfields F_2 and G_2 with $x \quad F$ containing F_1 and G_1, a function $f_2 : F_2 \leftrightarrow G_2$ extending f_1, and such that $val(F_2$

of $p(t)$ under f_1, then we know that $val(q(y)) = val(f(e_r))y^r$ because f_1 is an isomorphism. This proves $val(F_1(x)) = V$ and