On the Cauchy-Kowalevski theorem for analytic nonlinear partial differential equations

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April 12, 2018

1 Introduction

The name "Cauchy problem" is usually attributed to a class of boundary value problems associated to partial differential equations (PDE). The study of such problems began in earnest with Cauchy himself, who investigated the existence of solutions to analytic nonlinear PDE of the second order [1, 2, 3, 4, 5, 6]. This work was extended to general analytic nonlinear systems of PDE by Kowalevski in 1875 [11]. Both results are collectively known as the Cauchy-Kowalevski theorem, which is the primary focus of this paper. It is worth noting that in the same year as Kowalevski, Darboux published a similar result, which applied to less general problem [7]. In 1898, Goursat simplified Kowalevski's argument [10], and it is Goursat's proof that we present here.

The fact that properly defined Cauchy problems have unique analytic solutions is incredibly powerful. Equations such as the wave equation, Maxwell's equations, and the heat equation constitute Cauchy problems when paired with appropriate boundary conditions.

The applicability of the Cauchy-Kowalevski theorem is, however, limited. One major assumption for the theorem is that the functions describing the boundary data and the partial differential equation are all analytic (this term will be defined later). This is an unfortunately stringent requirement, and as

2 Tools

In this section we present the notation used throughout the argument for the

Care should be taken to avoid confusing *real analytic* functions and *complex analytic* functions, which are defined identically to real analytic functions with the word "real" replaced by "complex." This is because complex analytic func-

3 Cauchy-Kowalevski Theorem

The main objective of this section is the resolution of the Cauchy problem,

$$F(x, D^k u) = 0, \tag{1a}$$

$$\left.\frac{\partial^i u}{\partial n^i}\right|_S = \varphi_i, \quad 0 \le i \le k - 19.962600'$$

Consider the first order problem

$$\partial_t y_{\alpha i} = y_{\alpha(i+1)} \quad \text{for } 0 \le |\alpha| + i < k, \tag{3a}$$
$$\partial_t y_{\alpha i} = \partial_{x_j} y_{(\alpha-\sigma^j)(i+1)} \quad \text{for } |\alpha| + i = k, \quad i < k, \tag{3b}$$
$$\partial_t y_{\mathbf{0}k} = \frac{\partial G}{\partial t} + \sum_{|\alpha|+i<k} \frac{\partial G}{\partial y_{\alpha i}} y_{\alpha(i+1)} + \sum_{|\alpha|+i=k, i<k} \frac{\partial G}{\partial y_{\alpha i}} \partial_{x_j} y_{(\alpha-\sigma^j)(i+1)}, \tag{3c}$$
$$y_{\alpha i}(\mathrm{x}, 0) = \partial_{\mathbf{x}}^{\alpha} \psi_i(\mathrm{x}) \quad \text{for } i < k, \tag{3d}$$
$$y_{\mathbf{0}k}(\mathrm{x}, 0) = G(\mathrm{x}, 0, \psi), \tag{3e}$$

where, for each α, j is the smallest index with $\alpha_j = 0$, and G does not depend on $y_{\mathbf{0}k}$. We posit that if u is a solution of (2), then we can construct a solution to (3) by setting $y_{\alpha i} = \partial_{\mathbf{x}}^{\alpha} \partial_t^i u$. In fact, this can be readily seen by the construction of (3). A less obvious fact is that if y is a solution to (3), then $y_{\mathbf{0}0}$ is a solution to (2). We shall prove this now.

It can be clearly seen from (3a) that

$$y_{\alpha(i+l)} = \partial_t^l y_{\alpha i} \tag{4}$$

for $i + l \le k$. This fact and (3b) together imply that

$$\partial_t y_{\alpha i} = \partial_t \partial_{x_j} y_{(\alpha-\sigma^j)i} \tag{5}$$

for $|\alpha| + i = k$ and $i < k$. By integrating both sides of this equation with respect to t, we find that

$$y_{\alpha i}(\mathrm{x}, t) = \partial$$

for some function $f_{\alpha i} : \mathbb{R}^{n-1} \to \mathbb{R}$. But we can use the initial data to determine $f_{\alpha i}$. From (3d), we find

$$\begin{aligned} y_{\alpha i}(\mathrm{x}, 0) &= \partial_{\mathbf{x}}^{\alpha} \psi_i(\mathrm{x}) \\ &= \partial_{x_j} \partial_{\mathbf{x}}^{\alpha - \sigma^j} \psi_i(\mathrm{x}) \\ &= \partial_{x_j} y_{(\alpha - \sigma^j) i}(\mathrm{x}, 0), \end{aligned}$$

which implies that $f_{\alpha i}(\mathrm{x}) = 0$ everywhere. Hence (7) holds for all $\alpha = \mathbf{0}$.

Now we solve for $y_{\mathbf{0}k}$. From (3c), (4), and (7), we have

$$\begin{aligned} \partial_t y_{\mathbf{0}k} &= \frac{\partial G}{\partial t} + \sum_{|\alpha| + i < k} \frac{\partial G}{\partial y_{\alpha i}} y_{\alpha(i+1)} + \sum_{|\alpha| + i = k, i < k} \frac{\partial G}{\partial y_{\alpha i}} \partial_{x_j} y_{(\alpha - \sigma^j)(i+1)} \\ &= \frac{\partial G}{\partial t} + \sum_{|\alpha| + i \le k, i < k} \frac{\partial G}{\partial y_{\alpha i}} \partial_t y_{\alpha i} \\ &= \partial_t (G(\mathrm{x}, t, \end{aligned}$$

where Ψ is a vector-valued function containing the analytic functions ∂^{α}

where the

for all i

The solution to (20) is therefore given by

y (x

By taking a derivative with respect to $s = r^2$ and using Green's theorem, this reduces to

$$
\begin{aligned}
\frac{\partial V}{\partial s} &= \frac{1}{2r}\frac{\partial V}{\partial r} \\
&= \frac{1}{2r}\frac{\partial}{\partial r}\int_0^r \int_0^{2\pi} \left[\frac{\partial u}{\partial x}(\xi \cos\theta, \xi \sin\theta, t) + i\frac{\partial u}{\partial y}(\xi \cos\theta, \xi \sin\theta, t) \right] i\xi \, d\theta \, d\xi \\
&= \frac{1}{2r}\int_0^{2\pi} \frac{\partial u}{\partial x}(r \cos\theta, r \sin\theta, t
\end{aligned}
$$

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