Infinite Galois Theory

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1 Introduction

For an finite Galois extension E/F, the fundamental theorem of Galois Theory establishes an one-to-one correspondence between the intermediate fields of E/F and the subgroups of $Gal(E/F)$, the Galois group of the extension. With this correspondence, we can examine counts only the relevant subgroups.

2 Topological groups

Since we are going to put a topology on the infinite Galois groups. The concept of ”topological groups” will naturally arise. In this chapter, we are going to define the topological groups, and look into some properties of topological groups as well.

Definition 2.1. A group G is a topological group if it is a topological space, and the multiplication $\cdot : G \times G \to G$ is continuous, where GxG is equipped with product topology, and the inverse map $G \to G$: $g \to g^{-1}$ is continuous. A homomorphism between two topological groups is a continuous group homomorphism, and an isomorphism between two topological groups is a homeomorphic group isomorphism.

To understand the concept better, we should look at some examples:

Example 2.2. (1) Let G be the the group of all integers under the addition, where the topology on Z has discrete topology. Then $((Z), +)$ is a topological group.

(2)In fact, let G be any group, and equip it with the discrete topology, then it is a topological

Now we are going to look at some of the basic properties of the topological groups.

Proposition 2.3. *(Munkres' Chapter 18 Exercise 11) Let $F : X \times Y \to Z$. We say that F is continuous in each variable separately if for each y_0 in Y, the map $h : X \to Z$ defined by $h(x) = F(x \times y_0)$ is continuous, and for each x_0 in X, the map $k : Y \to Z$ defined by $k(x) = F(x_0 \times y)$ is continuous. If F is continuous, then F is continuous in each variable seperately.*

Proof. Let V be a open set in Z. Since $F : X \times Y \to Z$ is a continuous map. $F^{-1}(V)$ is an open set in $X \times Y$, denote it as $U_1 \times U_2$. Let $X \times \{y_0\}$ be the subspace of $X \times Y$. Then $F^{-1}(V) \cap X \times$

Proof. By the homogeneity and proposition 2.4, we only need to prove the proposition for the neighborhood containing the identity element e.

Consider the multiplication $\cdot : G \times G \to G$. Since U is a neighborhood of e, thus there exists an open set U containing e. Then the preimage of U is an open set in $G \times G$, which is of the form $V_1 \times V_2$, and V_1 and V_2 are both open sets in G. Let V

Proof. To show $\overline{H}$ is a subgroup of G, we only need to show that for every a, b $\overline{H}$, we have ab^{-1} $\overline{H}$.

Let W be a neighborhood of ab^{-1}. Since the multiplication is continuous, then there exists an open sets UV^{-1} W, where U is an open set contains a, and V is an open set contains b. Since a, b $\overline{H}$, then $U \cap H =$ and $V \cap H =$. Therefore, x $U \cap H$ and y $V \cap H$. Since H is a group, then y^{-1} $V^{-1} \cap H$ xy^{-1} $UV^{-1} \cap H$ $W \cap H$. This means the intersection of W and H is not empty. Since this is true for every neighborhood of ab^{-1}, (2)Let a $\overline{H}$. Since H is open, then since aH is a open set containing a aH is a neighborhood of a. Since a is in the closure of H, thus a is a limit point of H $aH \cap H =$. Then h_1, h_2 H, such that $ah_1 = h_2$ $aH \cap H$. $a = h_2 h_1^{-1}$. Since H is a subgroup, H is closed under multiplication and inverse mapping. Therefore, a H $\overline{H}$ H. Since H $\overline{H}$, we have $H = \overline{H}$. Hence, an open subset of $G \setminus \{a^{-1}b\}$. Then since the map $f : G \times G \to G$, where $(x, y) \to xy^{-1}$, is continuous, then open sets V, W in G such that VW^{-1} U. Thus, $a^{-1}b$ VW^{-1} $aV \cap bW =$. Hence G is Hausdorff. □

(1) $\varphi_{ik}(n) = \varphi_{jk} \circ \varphi_{ij}(n) = k$, if $k \leq n$;

and (2) $\varphi_{ik}(n) = \varphi_{jk} \circ \varphi_{ij}(n) = n$, if $n \leq k$.

Now we have done all the preparation, so we can define the profinite groups.

Definition 3.7. A *profinite group* G is a topological group that is isomorphic to a projective limit of a projective system $\{G_i, \varphi_{ij}\}_I$, where G_i are finite discrete topological groups $i \quad I$.

Just by the first glimpse of the profinite group's definition, it is naturally to wonder if a Galois group $G = Gal(E/F)$ of an infinite Galois extension E/F is a profinite groups, where the finite topological groups are G/H_i, and H_i are all the normal subgroups of G with finite index. In fact, this is true. Moreover, every profinite group is isomorphic to a Galois group.

4 Finite Galois theory and The Krull topology

In this Chapter, we will review some basics of the (finite) Galois theory, and also look into some properties of the Krull topology, which is the topology we are going to put on the infinite Galois groups.

Definition 4.1. The Galois extension is a algebraic field extension that is both normal and separable.

Definition 4.2. Given an field extension E/F, the Galois group $Gal(E/F)$ is the set of all automorphisms on the field E that fixed the field F (i.e, given a σ $Gal(E/F)$, $\sigma(x) = x$, if x F) under the composition.

Definition 4.3. Let $G = Gal(E/F)$, and H is a subgroup of G, the fixing field of H is denoted as $F($

(ii) K/F is normal if and only if H is a normal subgroup of G, and Gal(E/F) if H0nway

Proof. (1) To show G is Hausdorff, we need to show that for any two distinct elements $\sigma, \tau \in G$, $\exists$ neighborhood U of σ and neighborhood V of τ such that U and V are disjoint. Since $\bigcup_{K \in T} K = \bigcup_{U \in S} U = E$, and every automorphism in $Gal(E/K)$ fixes K, then every element in the intersection $\bigcap_{K \in T} Gal(E/K)$ fixes E.

$\bigcap_{K \in T} Gal(E/K) = \{e\}$

Since $Gal(K/F)$ is finite discrete space, Let $V_H = \bigcup_{x \in H \setminus U_H} xU_H$.Since U is open, then U_H is open all the cosets of U_H is open V_H is open.

Then (U_H, V_H) form a pair of separated sets of H. Since H is connected, and U_H is not an empty set,, then V_H is an empty set. Moreover, $U_H = U \cap H = U$. Since this is true for all U in the local basis. Then $H \subseteq \bigcap_{U \in S} U = \{id\}$. Hence the only connected component of $\{$

Let K be a splitting field over F, such that $F \quad K \quad E$ and $K \quad T$ and $\alpha \quad K$. Then $Gal(E/K)$ is an open subgroup of G.

Let $\tau \quad Gal(E/K)$. Then τ fixes K. $\quad \sigma\tau(\alpha) = \alpha$. $\quad \sigma Gal(E/K)$ is an open neighborhood of σ that does not intersect $Gal(E/F(H))$. Since this is true for every element outside $Gal(E/F(H))$. $\quad Gal$

Proof. Followed by the Lemma 5.1, $G(F(H) = \overline{H}$. Therefore, to make the bijection valid, H has to be closed.

(1) Since H_1, H_2 are closed subgroups of G, and

References

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