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YANG-MILLS THEORY ON A CYLINDER

Jean Weill

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1 Introduction

We can imagine that this complicated array of moving things which constitutes the world is something like a great chess game being played by the gods, and we are observers of the game. We do not know what the rules of the game are; all we are allowed to do is to watch the playing.

Richard Feynman (1918-1988)

Yang-Mills theory allows physicists to model particle dynamics and is at the core of the attempt to unify the forces in nature. More mathematically, Yang-Mills theory is a gauge theory for the groups SU(n). Noting that the electroweak force is described by U(1) × SU(2), and chromodynamics by SU(3), it's easy to see the power of a theory as general as Yang-Mills theory.

In this paper, we will study a toy model where Yang-Mills theory can be fully solved without any perturbation theory. Looking at simple cases is a good way to understand the theory better and can still be very useful practically. For example, solving Yang-Mills theory for a cylinder provides one with powerful tools to approach confinement in chromodynamics [4]. Now, let's get our hands dirty and see how to solve this not-so-simple toy model.

The Yang-Mills equations in the vacuum are given by [2]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{1}$$

$$\partial^\mu F_{\mu\nu} + [A^\mu, F_{\mu\nu}] = 0 \tag{2}$$

with

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

Note the antisymmetric property $F_{\mu\nu} = -F_{\mu\nu}$.

Let's look at the simplest case: one dimension of time and one of space. Then if μ or $\nu = 0, 1$, we get $F_{\mu\nu} = 0$. Thus we have no magnetic field. So 1 reduces to

$$F_{01} = \partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] \tag{3}$$

but $F_{01} = E_x = E$, thus 3 is equivalent to

$$E = \partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] \tag{4}$$

Doing the same thing with 2, we get the two equations

$$\frac{\partial E}{\partial t} + [A_0, E] = 0 \tag{5}$$

$$\frac{\partial E}{\partial x} + [A_1, E] = 0 \tag{6}$$

1.1 Infinite Minkowki Plane and Maxwell's Equations

A good check to make sure that what we're doing makes sense is to consider the abelian case where the commutators vanish. We should get the basic Maxwell's equations, i.e.,

$$\cdot E = \frac{\rho}{{}_0}$$
$$\times B - \frac{\partial E}{\partial t} = \mu_0 J$$
$$E = - \quad \varphi - \frac{\partial A}{\partial t}$$

but these simplify even more since we're considering the vacuum and the 1 + 1 dimensional case.

$$\frac{\partial E}{\partial x} = 0$$
$$\frac{\partial E}{\partial t} = 0$$
$$E = -\frac{\partial \varphi}{\partial x} - \frac{\partial A^1}{\partial t}$$

The Yang-Mills equations 4, 5, and 6 with vanishing commutators become

$$\frac{\partial E}{\partial x} = 0$$
$$\frac{\partial E}{\partial t} = 0$$
$$E = \frac{\partial A_1}{\partial t} - \frac{\partial A_0}{\partial x}$$

We seem a little bit o but that's not true. Recall that $A^\mu = (A^0, A^1) = (\varphi, A^1)$. We also have that $A_\mu = \eta_{\mu\nu} A^\nu$ with

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Thus, we get $A_\mu = (\varphi, -A^1) = (A_0, A_1) \rightarrow A_1 = -A^1$ and $A_0 = \varphi$. So we do indeed recover Maxwell's equations for the abelian case. We will let $A_1 = A$ for the rest of the paper.

The easiest non abelian case is to consider the infinite 2 dimensional Minkowski space, like we would work on the infinite Euclidian plane in electromagnetism. It turns out that this example is a bit too trivial. We want to work with a finite energy, but the electric field itself has energy given by

$$U = \frac{1}{2} \int_{\mathcal{V}} E^2 d\tau \tag{7}$$

Therefore in our case we can take $V \rightarrow \infty$. This means that in order to have a finite energy we must have $E \rightarrow 0$ at infinity. But from our equations, this means that E

1.2 Finite Minkoswki Cylinder and Wilson Loop

Recall that we want to find a gauge invariant quantity on a closed loop. In this case, 11 becomes

$$S \to g(x) S g^{-1}(x)$$

We want to take advantage of the fact that $g(x)g^{-1}(x) = I$. It follows that the trace of S is our desired gauge invariant quantity, called the Wilson loop.

We have

$$\mathrm{tr}(S) \to \mathrm{tr}\left(g(x) S g^{-1}(x)\right) = \mathrm{tr}\left(g^{-1}(x) g(x) S\right) = \mathrm{tr}(S)$$

Ta-da! We finally found a gauge invariant quantity on the loop. You can look at it this way:

- For a vector, the invariant quantity is length.
- For a group of vectors, it's the dot product.
- For a gauge field A, the information independent of the choice of the gauge transformation is the Wilson loop.

Note that S for any curve that is contractible to a point is equal to the identity, but not so for a curve that is non contractible. One more reason why S is perfect for us.

If you have studied Yang-Mills theory in the plane, you should recall that when you build the Yang-Mills theory Lagrangian you find that the invariant quantity that generates dynamics in the field is given by

$$\frac{1}{4}\,\mathrm{tr}(F^{\mu\nu} F_{\mu\nu})$$

Therefore, it shouldn't be hard to believe that $\mathrm{tr}(S)$ and $\mathrm{tr}(F$

2 Fixing the Gauge

There are two common strategies to solve the quantum theory:

- The first is to start by quantizing the field and then to fix the gauge. The di□culty with that technique is that quantizing first still leaves you with a infinite amount of degrees of freedom and is therefore unnecessary di□cult.
- The second is to start by fixing all the gauge and then to quantize the field. The problem with that strategy is due to Gribov who found that it's not possible to fix all the gauge, there is always a finite amount left.

We're going to find a compromise between these two ways by first fixing as much of the gauge as possible and then quantize the finite system that we have left. We will then use character functions and the Peter-Weyl theorem to fully solve the quantum theory and find the discrete global excitations in the field while still having some gauge unfixed.

Most of the argument is that the gauge group is infinitely dimensional but most of it can be fixed. Only a finite dimensional part can't be fixed, and we can work around this part using character functions.

2.1 Fixing All of the Time Gauge

In the time dimension, everything is topologically trivial. We can't impose any periodicity since then causality would be violated. It will not be that easy for the space dimension since it doesn't have to be true that $S(2\pi) = S(0)$. This is because there can be more than one curve connecting any two points and S depends not only on the endpoints but also on the curve itself. Therefore, for the time dimension, we should be able to fix all the gauge, but not for the space part.

How can we quantize the gauge field A? We know that all the gauge can be fixed expect a finite part. Therefore all the information should be contained in this little piece. We want to quantize the

It seems very likely that 12 should become 4. Note that we also know that all the time gauge

2.2 Fixing Most of the Space Gauge

To fix the space gauge, we want to work with the parallel transport S. It is well known that the S given by 10 is the solution to the following differential equation

$$\boxed{\frac{\partial S}{\partial x} = -A'S}$$

with $S(0) = 1$. Looking back at the first boxed equation and remembering that

$$(A')^1 = -(A')_1 = -A'$$

3 Quantizing the Finite Field

3.1 Hamiltonian Equations of Motion

Now, we want to express the Yang-Mills equation in term of the canonical variables q and p to then be able to quantize the field. These equations can be found by using the variational principle on the action N. Let's see what the action should be. From classical we have that

$$N = \int L dt$$

where $L = p\dot{q} - H$ is the Lagrangian. Note that for a non abelian group, we must do the following change

$$\dot{q} \to q^{-1}\dot{q}$$

This might seems weird but the explanation is the following. In the normal case, p transforms q like

$$q \to q + a$$

for some constant a, and $\dot{q} = p$ where we are off by a constant. For our system, we must have the slightly different

$$q \to bq$$

for some constant b. To get the normal action from this new non abelian one, we just need to replace q by $\ln(q)$ in order to transform multiplication into addition. Then,

$$p = \frac{d}{dt}\ln(q) = q^{-1}\dot{q} \tag{23}$$

where we again are off by a constant. Then we have

$$q' = \ln q \to \ln(bq) = \ln b + \ln q = a + q'$$

by letting $b = e^a$. Our action is now given by

$$N =$$

$$H = -\pi \operatorname{tr}\left(p^2\right)$$

Now we're back in business. We postulated that

$$N = \int \operatorname{tr}\left(pq^{-1}\dot{q}\right) dt + \int \pi \operatorname{tr}\left(p^2\right) dt$$

where we need to add a trace to the first term for the action to make any sense. Varying only p, we get

$$\begin{aligned}\delta N &= \int \operatorname{tr}\left((\delta p)q^{-1}\dot{q}\right) dt + \int 2\pi \operatorname{tr}(p\delta p) dt = \int \operatorname{tr}\left((\delta p)q^{-1}\dot{q} + 2\pi p\delta p\right) dt \\ &= \int \operatorname{tr}\left((q^{-1}\dot{q} + 2\pi p)\delta p\right) \\ &= 0 \rightarrow q^{-1}\dot{q} = -2\pi p\end{aligned}$$

Doing the same thing but only varying q we get these two equations

$$q^{-1}\dot{q} = -2\pi p \tag{24}$$

$$\frac{dp}{dt} = 0 \tag{25}$$

We can show that 24 and 25 contain the same amount of information as the Yang-Mills equations 16, 17, and 18.

Consider

$$\frac{\partial S}{\partial x} + A'S = 0$$

Taking the time derivative, we get

$$\frac{\partial \dot{S}}{\partial x} + \frac{\partial A'}{\partial t}S + A'\dot{S} = 0 \Leftrightarrow \partial \dot{S}$$

$$\hat{H} = \sum_i \hat{p}_i^2$$

Note that $\hat{p}_i$ is a first order derivative in space so $\hat{p}_i^2$ is a second order time derivative in space which is summed over the orthogonal basis. It must therefore means that the Hamiltonian is nothing less than our nice Casimir Operator we've already seen in our undergraduate quantum mechanics course when looking at angular momentum and spin! Thus, this seemingly complex problem has been reduced to a case of the Schroedinger equation we already know and so we have fully solved the problem.

For $G =$SU(2), we get that the irreducible representations are labeled by the spin

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, etc.$$

and the eigenvalues are given by

$$E_j = \pi j(j + 1)$$

4 Conclusion

The mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by Nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which Nature has chosen.

Paul Dirac (1902-1984)

In this paper, we have successfully fully solved the Yang-Mills equations on a cylinder by adopting a non conventional approach. We have showed that a system that might appear trivial, can still hide a lot of complexity. The system has no particles, no magnetic field, and a pretty easy topology but the field still has global excitations! This is unheard of in the normal Minkowski plane when one studies quantum field theory. For the trivial case, an excitation of the field means that you get particles. We found here that an excitation of the field is actually more general and doesn't have to produce particles for a non trivial topology.

A mind-blowing application of solving Yang-Mills theory on a cylinder has been discovered by Witten [3]. The idea is that you can use the cylinder trick to solve Yang-Mills theory on any Riemannian surface of arbitrary genus by splitting the surface into cylinders. Consider for example a Riemannian surface of genus 1. You just have to glue the ends of the cylinder and you're done. For a Riemannian surface of genus 2, you take 3 cylinders, put one in the middle and glue the other two to its ends. You can do this for an arbitrary genus!