Quandle Invariants of Knots and Links

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Abstract

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Definition 3. *Given two embeddings f, g from a manifold X into another manifold M, a continuous function $H : X \times [0,1] \longrightarrow M \times [0,1]$ is called an ambient isotopy if $(x,0) \mapsto f(x)$ and $(x,1) \mapsto g(x)$ where $H(x,t)$ is an embedding for all $t \in [0,1]$, and we say f and g are ambient isotopic.*

We say that two links are equivalent if there exists an ambient isotopy between them.

It would be difficult to study knots if we always had to work with these embeddings. The main way we get around this is through *knot diagrams*: Given a knot K, a knot diagram of K is a projection of it's image in $\mathbb{R}^3$ to a suitable plane: the projection must be bounded and have a finite numinclusion map $i : S^1 \to \mathbb{R}^3$. While the rightmost knot is the trefoil knot.

How can we show that two knots are the same? We may try to exhibit an ambient isotopy between either of the knots. But, beyond modeling the knot physically, it is difficult to show two knots are actually the same. Luckily, a theorem by Reidemeister [1] gives us an equivalent condition on knot diagrams to the existence of an ambient isotopy taking one knot to the other.

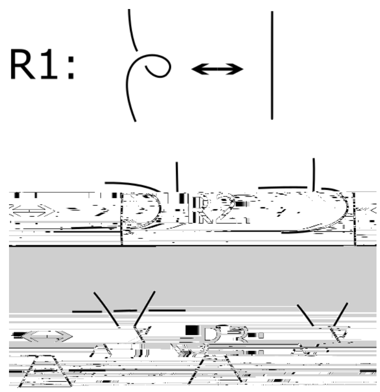

Figure 2: The Reidemeister Moves

Theorem 1. *For two links L and L', there is an ambient isotopy between them if and only if their diagrams are related by a finite sequence of moves in Figure 2 along with planar isotopies.*

The Reidemeister moves serve as a codification of the ways we maneuver a knot in three dimensional space. The first move adds a twist, the second move crosses one strand over/under another, and the third move passes a strand over/under a pre-established crossing. Using Reidemeister's theorem we are able to show that the complicated knot diagram in Figure 1 is actually the unknot, we do this in Figure 4.

An oriented knot is a knot along with a specified direction, this is typically signified by arrows along the knot diagram. There is an analog of Theorem 1 for oriented knots which can be found in [2]. Given an oriented knot we may reverse the orientation to obtain its *mirror image*. For some oriented knots it is possible to distinguish between mirror images — in this case we call the unoriented version of the knot *chiral* — the trefoil knot is the simplest example. One byproduct of adding an orientation is that there are now two types of crossings which we denote by left/right handedness. We do this because it is easy to determine what type of crossing you have by pointing your index- nger along the direction of the overstrand (palm down) and noting which hand has the thumb pointing along the outgoing understrand.

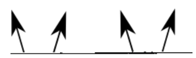

Figure 3: Left and Right Handedness

Figure 4: Unraveling a Complicated Unknot

Elementary Invariants

We've seen that the Reidemeister moves give a method to show equivalence of knots. But how do we know that the three knots in Figure 1 aren't actually all the same knot? This is a big problem since it's impossible to show that two knots are di erent using purely Reidemeister moves { it may be possible to simplify a knot by introducing some complexity.

A knot (resp. link) invariant is a function on the space of all knots which remains the same under ambient isotopy. For a function de ned on a knot diagram to be an invariant, by Theorem 1, it is equivalent that the function be invariant under the Reidemeister moves.

Geometric Invariants

An obvious way to begin distinguishing knots is to take a geometric quantity of a knot diagram and take the minimum over all possible diagrams of the knot. Some examples are:

- Crossing Number - The minimal number of crossings of any diagram.
- Uncrossing Number - The minimal number of crossing changes (overstrand becomes the understrand) needed to obtain the unknot or unlink.
- Genus - The minimal number of holes in a surface whose boundary is a knot K.
- Length - The minimum length of a knot or link if we give the strands a uniform thickness.

Fox n-coloring

Our next two invariants are classics. Here we present the interpretation by Ralph Fox [3]. Given a knot diagram, we color each of the arcs one of three colors such that at each crossing either all of the arcs are colored the same or are unique. A trivial coloring is one which uses a single color. A knot is *tricolorable* if there exists a non-trivial coloring of its knot diagram.

Theorem 2. *Tricolorability is a knot invariant.*

Proof. It su ces to show that tricolorability is conserved under the Reidemeister moves:

□

Figure 5: Visual Proof of Theorem 2

Since we know that tricolorability is an invariant we may nally be able to distinguish the knots in Figure 1. First we note that since the unknot can only be colored trivially, a tricolorable knot cannot be the unknot. Coloring where y is the label corresponding to the overstrand and $x; z$ are the labels for the two arcs of the understrand. By considering this condition over $\mathbb{Z}_n$

The knot group is generated by loops going around each arc. The *Wirtinger presentation* is the most common way to describe the knot group utilizing one of its diagrams. It has as its generators loops which go once around each arc of the diagram, and relations corresponding to each crossing.

The type of relation you get depends on the handedness of the crossing

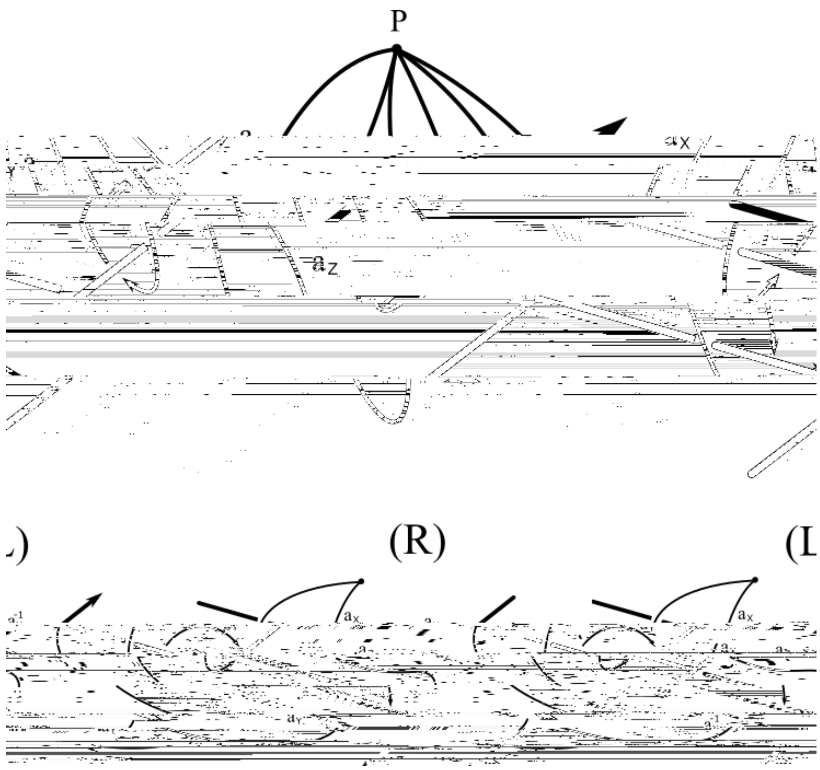

Figure 6: Wirtinger Presentation

$$a_x a_y^{-1} a_z a_y = 1 \quad (\text{or}) \quad a_z a_y a_x = a_y \qquad \text{(L)}$$

$$a_x a_y a_z a_y^{-1} = 1 \quad (\text{or}) \quad a_x a_y a_z = a_y \qquad \text{(R)}$$

The Wirtinger presentation of the knot group is then the free group on the generators modulo the smallest normal subgroup containing the set of relators of the form $a_x a_y^{-1} a_z a_y$ or $a_x a_y a_z a_y^{-1}$.

Let D_{2n} denote the dihedral group (the group of isometries of a regular n-gon). D_{2n} has a presentation: $D_{2n} = \{\alpha; s : \alpha^n = 1 = s^2; s\alpha s = \alpha^{-1}\}$. The rotations are the set $\{\alpha^k\}$ and is a cyclic subgroup of D_{2n} isomorphic to $\mathbb{Z}_n$. Reflections can all be written as $s_k := s\alpha^k$.

Theorem 4. *The set of Fox n-colorings of a knot K are in bijection with homomorphisms from the knot group's Wirtinger presentation to D_{2n}, which send the generators to reflections.*

Let $A = \{a_i\}$ be the arc set of a knot diagram of K. A Fox n-coloring is a map $C : A \to \mathbb{Z}_n$ which satisfies the condition $2y - x - z \equiv 0 \pmod{n}$ at each crossing. It is easy to verify that the mapping $a_k \mapsto s_{C(a_k)} \in D_{2n}$, determines a nontrivial homomorphism from the knot group to D_{2n}. Conversely, any nontrivial homomorphism arises in this way.

All the power of Fox n-colorings for knots follows from information in the knot group. Differentiating knots through presentations of their knot groups is a very difficult problem. As seen by Theorem 4 it is often easier to study maps emanating from the knot group rather than the knot group itself. This is a core idea for the invariants to come. However, to move forward we must gain a tool better at differentiating knots than the fundamental group.

Quandle Invariants

In order to improve the coloring invariants from last chapter we must realize a generalized version of our coloring set. With Fox n-colorings we took our colors to be elements of $\mathbb{Z}_n$ and then chose labels under the condition in Definition 4. After rearranging the equation we see that the label of an understrand is determined by the other two arcs. We will now consider what happens when we take a general set and impose an algebraic structure motivated by the Reidemeister moves.

Kei and Quandles

A Kei is a right-distributive groupoid which has the Reidemeister moves encoded in its structure. This is seen by first labeling each arc of a knot diagram by an element of a set X. We then say that if x is an understrand at a crossing, then overstrand y acts on x by right multiplication:

Figure 7: Kei Crossing Relation

Definition 6. *A Kei is a set X paired with a binary operation $\triangleright$ such that:*

(Idempotent) For all $x \in X$, $x \triangleright x = x$.

(Involutory) For all $x, y \in X$, $(x \triangleright y) \triangleright y = x$.

(Self-Distributive) For all $x, y, z \in X$, $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

The Kei axioms follow by assuming the crossing relation holds and then forcing the labeling to be invariant under the Reidemeister moves. The first and second kei axioms correspond the first and second Reidemeister moves respectively. The third axiom can be see from the following figure, by evaluating the label of the orange strand in two different ways:

FigXXB8: Kei Axiom 30 g 0 G 0 g 0 G Ee11.9552 Tf 4.552 0 Tdpo: Ke

See that every kei is a quandle (typically called involutory quandles) where B = B^{-1}. Additionally, the second axiom for quandles is equivalent to the right-action map β_y : Q ! Q being invertible for all y 2 Q. So we can actually just forget B^{-1} and consider a quandle to be a pair (Q; B) which satis es the above.

A function Φ : (Q_1; B$_1$) ! (Q_2; B$_2$) is a *quandle homomorphism* if

$$\Phi(x \text{ B}_1 y) = \Phi(a) \text{ B}_2 \Phi(b)$$

for all a; b 2 Q_1. The set of homomorphisms from Q_1 to Q_2 is denoted by Hom(Q_1; Q_2) and is equipped with a group structure via the composition operation.

The self-distributive property of quandles implies that β_y is a quandle homomorphism for every y 2 Q, and so is a quandle automorphism. We call each β_y the *point-symmetry about y* and the subgroup of $Aut(Q)$ generated by the point symmetries of Q is called the *inner automorphism group of Q* and is denoted by $Inn(Q)$.

Example 1. *Give 0 Tiphism:f307(a)-307(quandle)-307(u11.9552 Tf 9.271 0 Td [().)]TJ0 g 0 G/F4*

$$x \triangleright^{-1} y = z$$

where z is the outgoing understrand. Which relation we use depends on the type of crossing, as seen below.

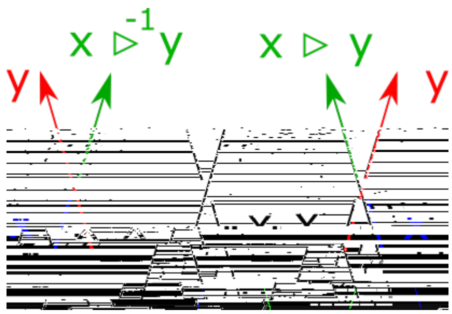

Figure 9: The Quandle Crossing Relation

Definition 9. *Given a diagram of a link L, the* **fundamental quandle** Q_L *is the free quandle on the arc-set A modulo the equivalence relations generated by the crossing relation.*

Theorem 5. *The fundamental quandle is a link invariant.*

Proof. We will show how the quandle axioms are motivated by the Reidemeister moves in such a way that the fundamental quandle is locally invariant.

R1: Going from one strand, labeled x, to a twist we know that two of the arcs must be labeled x. The other strand is $x \triangleright x$, so in order for it to be invariant we must have $x \triangleright x = x$ which follows from the first quandle axiom.

R2: Comparing the left and right sides of the R2 move, we require $y \triangleright z = x$. See that given any $z, x \in Q_L$ there should be a unique y such that $y \triangleright$

83 (b:9) 83 (b:9) 83 (b5 Td227 (e) -284 (a) -283 (unique)]TJ/F47 2n) -266 (the) -265 (ar) 5 11.9552 Tf 9.0

Example 5. *Here we will calculate the fundamental quandle of the oriented trefoil knot T. We start with three generators a, b, c, each corresponding to one of the arcs in Figure 1. The following choice is used only to exploit the three-fold symmetry of T. First choose an orientation, then label the strands as we traverse the knot so as to label them in reverse alphabetical order. We obtain the following crossing relations:*

$$a \triangleright b = c$$
$$b \triangleright c = a$$
$$c \triangleright a = b$$

Thus Q_T is partially given by the following operation table.

$\triangleright$	a	b	c
a	a	c	□
b	□	b	a
c	b	□	c

We get that Q_T is infinite and given by:

$$Q_T = \langle a, b, c \mid a \triangleright b = c, b \triangleright c = a, c \triangleright a = b \rangle$$

We now give a geometric description of the fundamental quandle of a knot (called knot quandle) adapted from [6]. Let K be an oriented knot in $\mathbb{R}^3$, and let $N(K)$ be a small tubular neighborhood about K, let $E(K) = \overline{(\mathbb{R}^3 \setminus N(K))}$. We let Π_K be the set of homotopy classes of paths in the space $E(K)$ with a fixed initial point, p, and endpoint on $\partial N(K)$. Let $m_y \subset E(K)$ be an oriented meridian of the tubular neighborhood hooking an arc, y, of the knot. Define $x \triangleright y = [\overline{x} \circ \overline{y^{-1}} \circ \overline{m_y} \circ \overline{y}]$, where $\overline{x}$ is a representative path of $x \in \Pi_K$ and we view each arc a as an element of Π_K where $\overline{a}$ is a path from p to a point on the boundary of the torus $\partial N(K)$ about the arc a and the path must travel only `over' the knot. The quandle axioms are easily checked, To see how Π_K is equivalent to Q_K from Definition 9, see Theorem 3.1 in [6].

The knot group acts naturally on the knot quandle. Fix a point p outside of the tubular neighborhood used as a basepoint for both the quandle and group. For a loop $\gamma \in \pi_1(K)$ and element δ of the quandle, $\gamma(\delta) = \delta \circ \gamma \in \Pi_K$. Furthermore, under this interpretation there is a natural map from the knot quandle to the knot group. For each element x of the knot quandle (a path from p to $\partial E(K)$) we may associate the loop $x^{-1} \circ m \circ x$, where m is the meridian passing through the endpoint of x. This shows that the knot group is assigned the color $x \in X$, then $a \mapsto x$. Furthermore, the map $C : Q_L \to X$ is a homomorphism. Take a crossing as in [9] where $C(a) = x$, $C(b) = z$, and $C(a \triangleright b) = y$, then since the crossing relation requires that $x \triangleright z = y$ we get that $C(a) \triangleright C(b) = C(a \triangleright b)$ for any two generators x, y of Q_L.

Theorem 6. *The Fox n-coloring invariant is related to $\Phi_X(L)$ where X is taken to be the dihedral quandle on n elements.*

Where our definition of Fox n-colorability was conveniently chosen to ignore trivial colorings obtained by using the same color throughout a diagram, the quandle coloring invariant (as a consequence of the first quandle axiom) does not differentiate. In other words, for any finite quandle X with $|X| = n$, $\mathrm{Hom}(Q_L, X)$ will have at least n elements corresponding to the constant maps. This is seen through the trefoil knot T, which has six Fox 3-colorings but $\Phi_{D_{2\cdot 3}}(T) = 9$. To see the latter simply note that homomorphisms are uniquely determined by where we send the generators of Q_T. By

See that Q_X is an enhancement of the quandle coloring invariant since we can recover $\Phi_X(L)$ from the cardinality of the vertex set of Q_X. By considering endomorphisms on X we are able to glean information about the structure of the coloring space. This is because the structure of the quandle quiver tells us if two elements of the fundamental quandle are related by an endomorphism on X. For examples of the quandle coloring quiver in action see Examples 5, 6, and 7 in [8]. One can also find certain polynomial invariants derived from the quandle coloring quiver, for this see [9, 11].

Definition 12. *A* **category**, $\mathcal{C}$, *is a class of objects* $\mathcal{O}$ *along with a set of maps between the objects called morphisms. Additionally a category must satisfy the following:*

1. *For each object* $a \in \mathcal{C}$ *there is an identity morphism* I_a *such that for any two morphisms* $f : a \to b$ *and* $g : c \to a$ *we have* $f \circ I_a = f$ *and* $I_a \circ g = g$.

2. *For any pair of morphisms* $f : a \to b$, $g : b \to c$, *there exists a composition morphism* $g \circ f : a \to c$, *and the composition of morphisms is associative.*

The quandle coloring invariant is a fairly useful, but it is integer valued and not functorial: the invariant does not associate anything to a map between spaces. The quandle coloring quiver is its *categorification*; for a fixed finite quandle X it associates each link to a set of vertices, and to every endomorphism of X a directed path on these vertices.

Theorem 8. *The quandle coloring quiver is a categorization of the quandle coloring invariant, with* X*-colorings of* L *as objects and elements of* $\mathrm{Hom}(X, X)$ *as morphisms.*

Proof. The identity map $I \in \mathrm{Hom}(X{:}X)$ satisfies the first axiom. Since composition of endomorphisms is an endomorphism, and composition is associative we are done. □

Quandle Cohomology

This section requires knowledge of homology and cohomology. For a primer see Appendix B. A **rack** is a quandle without the first (idempotent) axiom. For a finite quandle X, let $C_n^R(X)$ be the free abelian group generated by

$(x_1, \dots, x_n)$ for $x_i \in X$. The superscript "R" stands for rack. We define the boundary map:
$\partial_n : C_n^R(X) \to C_{n-1}^R(X)$ as the following:

$$\partial(x_1, \dots, x_n) := \sum_{i=2}^{n} (-1)^i [(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) - (x_1 \triangleright x_i, x_2 \triangleright x_i, \dots, x_{i-1} \triangleright x_i, x_{i+1}, \dots, x_n)]$$

for $n \geq 2$ and $\partial_n = 0$ for $n < 2$, and extend linearly. The chain complex is then:

$$\cdots \to C$$

For abelian groups A, B, C and homomorphism $f : B \to C$, we will let $\mathrm{Hom}_{\mathbb{Z}}(f, A) : \mathrm{Hom}(C, A) \to \mathrm{Hom}(B, A)$ be the homomorphism mapping $\phi \to \phi \circ f$ for all $\phi \in \mathrm{Hom}(C, A$

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Appendix A

The Fundamental Group

First formulated by Henri Poicare (April 29th, 1854 - July 17th, 1912), the fundamental group is a group associated to each topological space (in this paper we used a subset of $\mathbb{R}^3$). We first define *homotopy* as it allows use to define equivalence classes of functions. In particular: paths.

Definition 15. *Let X be a topological space and $x, y \in X$. A path from x to y is a continuous map $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Two paths α and β with endpoints $\alpha(0) = x = \beta(0)$ and $\alpha(1) = y = \beta(1)$ are called path homotopic ($\alpha \simeq \beta$) if there exists a continuous map $H : [0, 1] \times [0, 1] \to X$ which satisfies:*

$$\begin{aligned} H(s, 0) &= \alpha(s) \\ H(s, 1) &= \beta(s) \\ H(0, t) &= x \\ H(1, t) &= y \end{aligned}$$

One may think of H as a function along the space of paths in X where the endpoints are fixed. The time interval t is then a continuous deformation of path α to the path β. Thus path homotopy gives an equivalence relation on the set of paths in X from x to y. We denote $[\gamma]$ as the *homotopy class* containing the path γ. Thus $[\alpha] = [\beta] \iff \alpha \simeq \beta$. We also get an equivalence relation on elements of the set X. For $x, y \in X$ we say that they are path connected if there exists a path in X between x and y. For nice spaces (locally path connected), the path connected congruence classes correspond to the connected components of X.

We may define a binary operation, called *path composition*, between paths where the endpoint of one equals the intial point on the other.

Definition 16. *Let $x; y; z \in X$ and α be a path from x to y and β a path from y to z. Since $\alpha(1) = \beta(0)$ we can define:*

$$(\alpha\beta)(s) = \begin{cases} \alpha(2s); & 0 \leq s \leq \frac{1}{2} \\ \beta(2s - 1); & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Furthermore, this product is associative and can be extended to the equivalence classes of paths. If $\alpha_1(1) = \beta_1(0)$, $\alpha_1\beta_1 =$ $=$

Appendix B

Homology and Cohomology

Cohomology is one of the greatest contributions to mathematics of the last century. It is derived from homology, a powerful tool used as a Rosetta

$\partial_n : C_n \to C_{n-1}$ called *boundary maps*:

$$\ldots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_n(X) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

We require that composition of boundary maps is the constant map which sends all elements in $C_{n+1}(X)$ to the identity of $C_{n-1}(X)$:

$$\partial_n \circ \partial_{n+1} = 0_{n+1,n-1}$$

Or equivalently, $\mathrm{Im}(\partial_{n+1})$ $\ker(\partial_n)$. Furthermore, $\mathrm{Im}(\partial_{n+1})$ is a normal subgroup of $\ker(\partial_n$

The n^{th}-cohomology group, $H^n_A(X)$, is then the *n-cocylces* ker(δ^n