ON THE DISCRIMINANT OF THE HECKE RING, T_k, AND ITS INDEX IN THE RING OF INTEGERS OF T_k Q

1.

$$= \frac{a(a'z + d') + b(c'z + d')}{c(a'z + b') + d(c'z + d')}$$

$= (aa' + bc')z + (ab' + bd$ z [(ab)]TJ/F13 6.9738 9.9626 Tf 3.8740

Example 2.13. *Examples of weakly modular functions include constant functions and Eisenstein series, which are defined in Definition 2.18.*

Corollary 2.14. *The only weakly modular function of odd weight is the zero function.*

Proof. Let $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $f(z)$ a weakly modular function of odd weight k. Using Definition 2.12 and that $Az = \frac{-z+0}{0-1} = z$,

$$\begin{aligned} f(z) &= f(Az) \\ &= f\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} z\right) \\ &= (-1)^k f(z). \end{aligned}$$

Clearly, if k is odd, then $f(z) = 0$.

Let f be a weakly modular function. By (2) in the equivalent characterization of weakly modular functions in Definition 2.12, $f(z+1) = f(z); \forall z \in H$. Because of this, f is equal to some function $g(q)$ where $q = e^{2\pi iz}$ and if f is holomorphic, then $g(q)$ is holomorphic on the unit disk minus the origin. Using the equality $|q| = e^{-2\pi Im(z)}$, we see that $q \to 0$ if and only if $Im(z) \to \infty$ (the previous paragraph is due to [2, pg. 3]).

Thus, when f extends meromorphically (holomorphically) function at the origin, we say it is meromorphic (holomorphic) at infinity. By "extends meromorphically (holomorphically) at the origin," we mean if there exists some meromorphic (holomorphic) funtion h on the unit disk such that $h(z) = g(q)$ on the unit disk minus the origin.

Definition 2.15. *Let $k \in \mathbb{Z}$ and f a weakly modular function. f is called modular if f is holomorphic on H and at infinity, where we consider infinity to lie far in the imaginary direction.*

With this, one can characterize a modular form of weight k as a series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-p)^n; \quad a_i \in \mathbb{C} \tag{2.6}$$

for all $p \in H$, and supposing the second condition from the equivalent characterization of weakly modular in Definition 2.12 is satisfied, one can write $f(z)$ as a function of $q = e^{2\pi iz}$. Thus, a modular form of weight k is given by

$$f(z) = \sum_{n=0}^{\infty} a_n q^n = \sum_{n=0}^{\infty} a_n e^{2\pi niz}; \tag{2.7}$$

which converges absolutely for $|q| < 1$.

Definition 2.16. [2, Definition 1.1.3] *A modular form is called a cusp form if $a_0 = 0$ in its q-expansion; equivalently, a modular form is a cusp form if $\lim_{Im(z) \to \infty} f(z) = 0$.*

It's well known that the space of modular forms of weight k and the space of cusp forms of weight k over the full modular group (commonly denoted $M_k(SL_2(\mathbb{Z}))$ and $S_k(SL_2(\mathbb{Z}))$ respectively) are vector spaces over $\mathbb{C}$, and that $S_k(SL_2(\mathbb{Z}))$ is a subspace of $M_k(SL_2(\mathbb{Z}))$. One could also characterize the space of cusp forms of weight k as the kernel of the map $\Phi : M_k(SL_2(\mathbb{Z})) \to \mathbb{C}$ by $\Phi : \sum_{n=0}^{\infty} a_n q^n \mapsto a_0$.

Remark 2.17. As in [2, pg. 4], one typically denotes the space of modular forms

$$M(SL_2(\mathbb{Z})) = \bigoplus_{k \in \mathbb{Z}} M_k(SL_2(\mathbb{Z}));$$

which is a graded ring (the product of two modular forms of weight k and weight k' modular forms is a form of weight $k + k'$).

In addition, the space of cusp forms

$$S(SL_2(\mathbb{Z})) = \bigoplus_{k=0}^{\infty} S_k(SL_2(\mathbb{Z}))$$

forms a graded ideal in $M(SL_2(\mathbb{Z}))$ ([2, pg. 6]).

De nition 2.18. *Let $k > 2$. The function*

$$G_k(z) = \sum_{(c;d) \in \mathbb{Z}^2 \setminus (0;0)} \frac{1}{(cz + d)^k} \tag{2.8}$$

is called the Eisenstein series of weight k, where $G_k(\infty) = 2\zeta(k)$, where ζ denotes the Riemann zeta function given by $\zeta(k) = \sum_{d=1}^{\infty} 1/d^k$ (see [7, Proposition 4]*).*

Fact 2.19. [2, pg. 5] The Eisenstein series of weight k for all $k \in \mathbb{Z}_{\geq 3}$ is a modular form of weight k, and if one writes it in its q-expansion,

$$G_k(z) = \sum_{n=0}^{\infty} a_n q^n;$$

with $q = e^{2\pi i z}$, then $G_k(0) = 2\zeta(k)$ where ζ is the Riemann zeta function.

The Eisenstein series, G_k, is commonly normalized in two di erent ways: the rst normalizes the constant term and the second normalizes the coe cient of q in the q-expansion for G_k. The former will is denoted E_k and the latter is denoted G_k.

The normalized Eisenstein series of weight k, G_k, can be expressed in the following way:

$$G_k(z) = \frac{1}{2}\zeta(1 - k) + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \tag{2.9}$$

where $\sigma_{k-1}(n) = \sum_{m|}$

-expansion,P

c;d k-1(n) 260.491 Td [(3) TJ /F8 9.9626 Tf 1. -833 Td [(Tf 3.875 0 Td [

where $\sigma_{k-1}(n) = \sum_{\substack{m \mid n \\ m>0}} m^{k-1}$ and B_k represents the k^{th} Bernoulli number. This expression for E_k uses the identity $\frac{2k}{B_k} = \frac{-2}{\zeta(1-k)}$.

Defining the forms $f_1(z) = 60G_4(z)$ and $f_2(z) = 140G_6(z)$, one arrives at what is commonly known as the discriminant function, $\Delta : H \to \mathbb{C}$, given by $\Delta(z) = (f_1(z))^3 - 27(f_2(z))^2$ ([2, pg. 6]) which is a modular form of weight 12 (as $(f_1(z))^3$ and $f_2(z))^2$ are forms of weight 12). It is easy to check that the first term in the q-expansion for Δ is zero, and so by Definition 2.16, we conclude that $\Delta(z)$ is a cusp form. We verify in the proof of Theorem 2.21 that Δ is not the zero function and is zero nowhere except at infinity.

It is often useful normalize the coefficient of q in the q-expansion of Δ, which we will denote as Δ and is described in the following way: $\Delta(z) = (1/1728)(E_4^3 - E_6^2)$. Since E_4 and E_6 have only rational coefficients, it follows that Δ does as well.

Defining Δ allows one to develop another common modular function, $j : H \to \mathbb{C}$ given by $j(z) = 1728\frac{(f_1(z))^3}{\Delta(z)}$. The j function is known as the modular invariant since $j(Az) = j(z); \forall A \in SL_2(\mathbb{Z})$ ([[2]]). Since the only zero of Δ is at infinity, one observes that j has a simple pole at infinity (which shows why it is not a modular form).

Lemma 2.20 ([7, Theorem 3]). *Some notation from the theorem in* [7] *is used. Let $p \in H$, let f be a modular form, and let G denote the full modular group. Let $ord_p(f)$ be the integer s for which $f/(z-p)^s$ is nonzero. If f is a nonzero modular form of weight k, then the following formula is satisfied:*

$$ord_{\infty}(f) + \frac{1}{2}ord_i(f) + \frac{1}{3}ord_{\rho_3}(f) + \sum_{p \in H/G}^{*} ord_p(f) = k/12 \qquad (2.11)$$

where $\rho_3 = e^{2\pi i/3}$ is a third root of unity, and where $\sum^{*}$ means to take p not in the equivalence classes of neither i nor ρ_3.

Note that as f is a modular form, it has no poles. In particular, $ord_{\infty}(f); ord_p(f) \geq 0; \forall p \in H/G$.

Theorem 2.21. [7, Theorem 4]
(1) If $k < 0$ or positive and odd, or if $k = 2$, then $M_k(SL_2(\mathbb{Z})) = \{0\}$.
(2) Multiplication by Δ gives an isomorphism between $M_{k-12}(SL_2(\mathbb{Z}))$ and $S_k(SL_2(\mathbb{Z}))$.

Proof. The following proof is due to [7], and we will be using some of their notation. Let f be a modular form of weight k, and again let G denote the full modular group. Since the left hand side of Formula 2.11 is nonnegative for modular forms, k must be nonnegative and hence the only modular forms of negative weight are the zero function.

If k is positive and odd, Corollary 2.14 showed that the only forms satisfying this are also the zero function.

If $k = 2$, then the righ ... ormula 2.11 equals $\frac{1}{6}$. M ... le by 6 gives:

$$6ord_\infty(f) ... + 2ord_{\rho}(f) + 6 \sum_{p \in H/G} ord_p(f) = 1.$$

But $ord_\infty(f); ord_p(f) \in \mathbb{Z}_{>0}; \forall p \in H/G$, thus giving us a sum of nonn... tegers equal to 1, which is impossible. Thus, any modular form of weig... zero function. This proves (1).

For the sake of brevity, let $a = ord_\infty(f); b = ord_i(f)$, and $c = or$... that the discriminant function $\Delta = (60G_4(z))^3 - 27(140G_6(z))^2$. S... G_6 are modular forms, they satisfy Formula 2.11. Moreover, letting ... akes the right hand side of Formula 2.11 an element of $\mathbb{Q} \setminus \mathbb{Z}$ and s... $rd_p(f)$ must be zero. Applying the Formula to G_4 and multiplyi... , we have

$$6a + 3b + 2c = 2.$$

...early, the only solution is $(a; b; c) = (0; 0; 1)$.

...imilarly, applying the formula to G_6 and ... ough by 6, we have

$$6a + 3b ...$$

..., the only solution is $(a; b; c)$ ... gether, this tells us G_4 has one ... $_3$ and G_6 has one zero at ... nnot be the zero function because ...t zero at i. We've alread... is a cusp form of weight 12, and so ...ng Formula 2.11 to

$$... \sum_{p \in H/G} ord_p(f) = 1$$

...plies that $b = c =$ ... $d_p(f) = 0$ and proves that Δ is nonzero on H except at infinity (in...

Let $h \in S_k(SL_2(\mathbb{Z}$ ... infinity. Since Δ has a simple zero at infinity and nowhere else, g is holomorphic on H and at infinity. Clearly, g has weight $k - 12$. Thus, $g \in M_{k-12}(SL_2(\mathbb{Z}))$ and (2) is proven.

Corollary 2.22. *If $k = 0; 4; 6; 8; 10$, then the space of modular forms of weight k is one dimensional with gener98* respectively shows that $dim(M_k(SL_2(Z))) = 1$ and such nonzero forms generate

Proof. By Theorem 2.23, for any $k \in \mathbb{Z}_{>0}$, $M_k(\mathrm{SL}_2(\mathbb{Z}))$ has a basis $\{G_4^a G_6^b\}$ with $4a + 6b = k$. By construction, $M(\mathrm{SL}_2(\mathbb{Z})) = \bigoplus_{k=0}^{\infty} M_k(\mathrm{SL}_2(\mathbb{Z}))$, so G_4 and G_6 certainly generate $M(\mathrm{SL}_2(\mathbb{Z}))$.

Corollary 2.25. [7, Corollary 1] *Let $k > 0$ even. The dimension of $M_k(\mathrm{SL}_2(\mathbb{Z}))$ can be computed as follows:*

$$dim(M_k(\mathrm{SL}_2(\mathbb{Z}))) = \begin{cases} \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2 \bmod 12 \\ \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv 2 \bmod 12 \end{cases}$$

Proof. The following proof is due to [2]. This is obviously true for even k with $0 \leq k < 12$. Note that when $k > 2$ and even, $M_k(\mathrm{SL}_2(\mathbb{Z})) \neq \{0\}$, so the map in Corollary 2.22 is onto.

Theorem 2.21 proved that $M_k(\mathrm{SL}_2(\mathbb{Z})) \cong S_{k+12}(\mathrm{SL}_2(\mathbb{Z}))$, and the proof of Corollary 2.22 yielded that $dim(S_{k+12}(\mathrm{SL}_2(\mathbb{Z}))) = dim(M_{k+12}(\mathrm{SL}_2(\mathbb{Z}))) - 1$. Thus, $dim(M_{k+12}(\mathrm{SL}_2(\mathbb{Z}))) = dim(M_k(\mathrm{SL}_2(\mathbb{Z}))) + 1$. Replacing k by $k + 12$ in the above formula yields the same.

Although in Corollary 2.24 we only have a result about the basis for $M_k(\mathrm{SL}_2(\mathbb{Z}))$, Theorem 4 in Chapter 10 of [6] gives a more general result about the basis of $M_k(\mathrm{SL}_2(\mathbb{Z}))$, produced below (omitting the proof).

Theorem 2.26. [6, Chapter 10, Theorem 4] *A basis β for $M_k(\mathrm{SL}_2(\mathbb{Z}))$ with coefficients in $\mathbb{Z}$ (which is also a basis for $M_k(\mathrm{SL}_2(\mathbb{Z}))$ with coefficients in $\mathbb{C}$) is:*

(1) If $k \equiv 0 \bmod 4$, then $\beta = \{E_4^a \Delta^b\}$ with $4a + 12b = k$.

(2) If $k \equiv 2 \bmod 4$, then $\beta = \{E_6 E_4^a \Delta^b\}$ with $4a + 12b = k - 6$.

3. Congruence Subgroups

Definition 3.1. [2, pg. 13] *Let $n \in \mathbb{Z}^{+}$* 8 9F5 Td [(2696267(SL2 ...

Proof. We follow the proof in [2, pg. 13], and we will show successive normality separately: that is, we first show normality of $\Gamma(n)$ in $\Gamma_1(n)$, then normality of $\Gamma_1(n)$ in $\Gamma_0(n)$.

Define a map

$$\varphi : \Gamma_1(n) \to \mathbb{Z}_n$$

where for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(n)$,

$$\varphi(A) = b \,(\text{mod n}).$$

We have,

$$\begin{aligned} ker(\varphi) &= \{A \in \Gamma_1(n) \mid \varphi(A) \equiv 0 \text{ (mod n)}\} \\ &= \{A \in \Gamma_1(n) \mid b \equiv 0 \text{ (mod n)}\} \\ &= \Gamma(n). \end{aligned}$$

Since $ker(\varphi)$ is always a normal subgroup of $\Gamma_1(n)$, this shows that $\Gamma(n) \trianglelefteq \Gamma_1(n)$.

Define another map

$$\varphi' : \Gamma_0(n) \to (\mathbb{Z}_n)$$

where for any $B = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma_0(n);$

$$\varphi'(B) = d_1 \text{ (mod n)}.$$

We have,

$$\begin{aligned} ker(\varphi') &= \{B \in \Gamma_0(n) \mid \varphi'(A) \equiv 1 \text{ (mod n)}\} \\ &= \{B \in \Gamma_0(n) \mid d_1 \equiv 1 \text{ (mod n)}\} \end{aligned}$$

By definition of $SL_2(\mathbb{Z})$; $det(B) = 1$, so $ad = 1$, implying that $ad \equiv 1$ (mod n). If we require that $d \equiv 1$ (mod n), then $a \equiv 1$ (mod n) as well. Then $ker(\varphi')$ is exactly $\Gamma_1(n)$. Again, as $ker(\varphi')$ is normal in $\Gamma_0(n)$, we have shown $\Gamma_1(n) \trianglelefteq \Gamma_0(n)$.

Corollary 3.4. [2, pg. 14] *Let $\Gamma(n)$; $\Gamma_0(n)$; $\Gamma_1(n)$ be defined as in Equations (3.1), (3.2), (3.3) respectively. Then,* $[\Gamma_1(n) : \Gamma(n)] = n$; $[\Gamma_0(n) : \Gamma_1(n)] = \varphi(n)$, *where $\varphi(n)$ is the Euler totient Function.*

Proof. The maps defined in Proposition (3.3), φ and φ', are clearly onto maps. Using the First Isomorphism Theorem together with Proposition (3.3), we have that $\Gamma_1(n)/\Gamma(n) = \mathbb{Z}_n$ which implies that $|\Gamma_1(n)/\Gamma(n)| = [\Gamma_1(n) : \Gamma(n)] = |\mathbb{Z}_n| = n$. Similarly, we have that $|\Gamma_0(n)/\Gamma_1(n)| = [\Gamma_0(n) : \Gamma_1(n)] = |(\mathbb{Z}_n)| = \varphi(n)$.

Definition 3.5. [2, pg. 164] *Let* $\Gamma_1; \Gamma_2$

Definition 3.6. [2, pg. 14] *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$*. The factor of automorphy* $j(A;z) \in \mathbb{C}$ *for* $z \in \mathbb{H}$ *is given by*

$$j(A;z) = cz + d.$$

Define the weight k *operator,* $f[A]_k$*, on functions* $f : \mathbb{H} \to \mathbb{C}$ *by*

$$(f[A]_k)(z) = j(A;z)^{-k} f(Az); \ A \in \mathrm{SL}_2(\mathbb{Z}). \tag{3.4}$$

The previously defined weight k operator $f[A]_k$ can be generalized to matrices $B \in \mathrm{GL}_2^+(\mathbb{Q})$ by $(f[B]_k)(z) = det(B)^{k-1} j(B;z)^{-k} f(Bz); \ A \in \mathrm{SL}_2(\mathbb{Z}).$ This is in fact a generalization since this reduces to our definition when using matrices in $\mathrm{SL}_2(\mathbb{Z})$ (by definition, they always have determinant equal to 1).

Remark 3.7*.* In Definition 3.6, note that the functions on which $f[A]_k$ operates are not necessarily weakly modular; we can actually use the previously defined operator $f[A]_k$ on functions $f : \mathbb{H} \to \mathbb{C}$ to form an equivalent definition of weakly modular: a function is weakly modular of weight k if $f[A]_k \equiv f; \forall A \in \mathrm{SL}_2(\mathbb{Z}).$ This "new" definition clearly coincides with Definition 2.12.

Definition 3.8. [2, pg. 165] *Let* $\Gamma_1; \Gamma_2$ *be congruence subgroups of* $\mathrm{SL}_2(\mathbb{Z})$*, and let* $A \in \mathrm{GL}_2^+(\mathbb{Q})$*. Define the weight* k *operator* $[\Gamma_1 A \Gamma_2]_k$ *on functions* $f : \mathbb{H} \to \mathbb{C}$ *by*

$$f[\Gamma_1 A \Gamma_2]_k = \sum_j f[\beta_j]_k \tag{3.5}$$

where the β_j *are orbit representatives from the action of* Γ_1 *on* $\Gamma_1 A \Gamma_2$ *(see below Definition* 3.5*).*

Definition 3.9. [2, Definition 1.2.3] *Let* Γ *be a congruence subgroup and let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$*. We say a function* $f : \mathbb{H} \to \mathbb{C}$ *is a modular form of weight* k *with respect to* Γ *if the following three properties hold:*

1. f is holomorphic on $\mathbb{H}$.
2. $f(Az) = (cz+d)^k f(z); \forall A \in \Gamma.$
3. $f[A]_k$ is holomorphic at infinity $\forall A \in \mathrm{SL}_2(\mathbb{Z}).$

As in [2, pg. 17], we denote the space of modular forms of weight k with respect to Γ as $M_k(\Gamma)$. Moreover, $S_k(\Gamma)$ are the cusp forms of weight k with respect to Γ. The space of modular forms with respect to Γ is the set

$$M(\Gamma) = \bigoplus_{k=0}^{\infty} M_k(\Gamma)$$

which forms a graded ring. The set of cusp forms with respect to Γ

$$S(\Gamma) = \bigoplus_{k=0}^{\infty} S_k(\Gamma)$$

is a graded ideal in $M(\Gamma)$.

4. Hecke Operators

Definition 4.1. *Consider the double coset given by* $\Gamma_1(n)\alpha\Gamma_1(n)$*, where* $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ *and* p *is a prime.*

The Hecke operator, $T_p : M(\Gamma_1(n)) \to M(\Gamma_1(n))$*, is given by*

$$\begin{aligned} T_p(f)(z) &= f[\Gamma_1(n)\alpha\Gamma_1(n)]_k(z) \\ &= \sum_j f[\alpha_j]_k \end{aligned}$$

where α_j *are distinct orbit representatives.*

Lemma 4.2. [2, Proposition 5.2.1]

$$T_p(f) = \begin{cases} \sum_{j=0}^{p-1} f\left[\begin{bmatrix} 1 & j \\ 0 & p \end{bmatrix}\right]_k; & \text{if } p \mid n \\ \sum_{j=0}^{p-1} f\left[\begin{bmatrix} 1 & j \\ 0 & p \end{bmatrix}\right]_k + f\left[\begin{bmatrix} a & b \\ n & p \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}\right]_k; & \text{if } p \nmid n; \text{ where } ap - nb = 1. \end{cases}$$

Proposition 4.3. *Let* $f(z) = \sum_{j=0}^{\infty} a_m q^m \in M(\Gamma_1(n))$ *and let* p *be a prime not dividing* n*. The effect of the Hecke operator* T_p *on* $f(z)$ *can be characterized as*

$$T_p : \sum_{m=0}^{\infty} a_m q^m \mapsto \sum_{m=0}^{\infty} a_{mp} q^m + p^{k-1} \sum_{m=0}^{\infty} a_m q^{mp}$$

Proof.

5. Modular Forms mod ℓ

Definition 5.1. *Let* ℓ *be a prime and let* v_ℓ *be the* ℓ*-adic valuation of* $\mathbb{Q}$*. That is, if for any* $a \in \mathbb{Q}$ *one writes* $a = \ell^t(b/c)$ *where* $t \in \mathbb{Z}$ *and* ℓ *divides neither* b *nor* c*. Then,*

$$v_\ell(a) = t.$$

If an element a *of* $\mathbb{Q}$ *has nonnegative* ℓ*-adic valuation,* a *is said to be* ℓ*-integral.*

If $f(z) \in \mathbb{Q}[[$

This implies $f(z) \in \widetilde{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ is a sum of powers of Δ with coefficients in $\mathbb{F}_\ell$. Equivalently, this means $\widetilde{M}_k(\mathrm{SL}_2(\mathbb{Z})) \subseteq \mathbb{F}_\ell[\Delta/(2\pi)^{12}]$. The reserve containment is obvious after noting that Δ has integer (hence ℓ-integral) coefficients.

From this point, unless otherwise stated, we will work in level 1 (i.e. $\Gamma = \mathrm{SL}_2(\mathbb{Z})$) and assume that $\ell > 3$ is prime, since the by the previous proposition, the cases of $\ell = 2$ or 3 are trivial.

We will now introduce three different operators on the space of modular forms mod ℓ. The first operator is Atkin's U_ℓ operator (as in [3, pg. 255]), and it takes $\widetilde{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ to itself. We denote it by U. For each of the following operators, we describe their effect of a modular form $f(z)$ by describing its effect on the q-expansion of $f(z)$. So, let $f(z) = \sum_{n=0}^{\infty} a_n q^n \in \widetilde{M}(\mathrm{SL}_2(\mathbb{Z}))$.

$$(i)\ U : \sum_{n=0} a_n q^n \mapsto \sum_{n=0} a_{nl} q^n$$

$$(ii)\ V : \sum_{n=0} a_n q^n \mapsto \sum_{n=0} a_n q^{nl}$$

$$(iii)\ \theta : \sum_{n=0} a_n q^n \mapsto \sum_{n=0} n a_n q^n$$

Proposition 5.5. [3, Fact 2.2] *The following describe some of the relationships between the previously introduced operators.*

$$(i)\ f|VU = f; \quad \forall f \in \widetilde{M}_k(\mathrm{SL}_2(\mathbb{Z}));$$
$$(ii)\ ker(\theta) = Im(V);$$
$$(iii)\ Im(\theta) = ker(U).$$

Proof.

Fact 5.6. [4, Fact 1.7] If $f \in \widetilde{M}_k(\mathrm{SL}_2(\mathbb{Z}))$, then $w(f|V) = \ell w(f)$.

Fact 5.7. [4, Fact 1.4] The operator θ maps $\widetilde{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ to $\widetilde{M}_{k+\ell+1}(\mathrm{SL}_2(\mathbb{Z}))$. In particular, $w(f|\theta) \leq w(f) + \ell + 1$.

Lemma 5.8. [4, Lemma 1.9] *Let* $f \in \widetilde{M}_k(\mathrm{SL}_2(\mathbb{Z}))$. *Then* $w(f|U) \leq (w(f) - 1)/\ell + \ell$.

Proof.

Definition 5.9. [4, Definition 2.1] *The set* $\{\lambda_p\}$ *is called a system of eigenvalues if there is some nonzero eigenform* f *such that* $f|T_p = \lambda_p f$, *for all primes* p.

Definition 5.10. [4, Section 3] *Let* $\mathbb{T}_k$ *be the subring of* $End_{\mathbb{C}}(M_k(\mathrm{SL}_2(\mathbb{Z})))$ *generated by the Hecke Operators. The subring* $\mathbb{T}_k$ *is called the Hecke ring.*

Definition 5.11. [4, Section 3] *Let* R_k *be the subring of* $End_{\mathbb{F}_\ell}(\widetilde{M}_k(\mathrm{SL}_2(\mathbb{Z})))$ *generated by the Hecke Operators. The subring* R_k *is called the Hecke ring mod* ℓ.

Remark 5.12. The above two definitions can be generalized to any level n congruence subgroup, but for the purposes of this paper and results presented, we are limiting ourselves to level 1.

The ring $R_k = \mathbb{T}_k/\ell\mathbb{T}_k$ is an Artin ring, hence the ring $R_k \otimes \overline{\mathbb{F}}_\ell$ is also an Artin ring. As such, $R_k \otimes \overline{\mathbb{F}}_\ell$ has a finite number of maximal ideals and can be decomposed into a finite direct product of local Artin rings, which w.907 6.7kiuilal

$$\ell + 1 < w(f) \leq \begin{cases} 2\ell & \ell \geq 13 \text{ and } k \geq 2\ell^2 \\ 3\ell - 1 & \ell = 7 \text{ or } 11 \text{ and } k \geq 3\ell^2 - \ell \\ 3\ell + 3 & \ell = 5 \text{ and } k \geq 3\ell^2 + 3\ell \end{cases}$$

Lemma 5.17. [4, Lemma 4.2] *Let $S_k^j \subset M_k(\widehat{SL_2(\mathbb{Z})}) \otimes \overline{\mathbb{F}_\ell}$ be the generalized eigenspace associated to the local component A_k^j in $R_k \otimes \overline{\mathbb{F}_\ell}$. Let m_j be the unique maximal ideal in A_k^j. If S_k^j has a form f of filtration s with $k/\ell \geq s \geq \ell + 1$, then $\dim_{A_k^j/m_j}(m_j/m_j^2) \geq 2$.*

Proof.

Theorem 5.18. *If we are in any of the following cases*

$$1.\ \ell \geq 13 \text{ and } k \geq 2\ell^2;$$
$$2.\ \ell = 7 \text{ or } 11 \text{ and } k \geq 3\ell^2 - \ell;$$
$$3.\ \ell = 5 \text{ and } k \geq 3\ell^2 + 3\ell$$

then $\dim_{A_k^j/m_j}(m_j/m_j^2) \geq 2$ for at least one $j \in \{1, 2, \ldots, n\}$.

Proof. Assume we are in one of the cases listed in the Theorem. Then, by Fact 5.16, at least one of the generalized eigenspaces of $R_k \otimes \overline{\mathbb{F}_\ell}$ must have a form f of filtration satisfying $\ell + 1 < w(f) \leq k/\ell$. Then by the lemma, the associated local component(s) to said generalized eigenspace(s) must have Zariski tangent dimension at least 2.

Proposition 5.19. *Any S_k^j contains a simultaneous eigenform f of filtration $w(f)$ such that $w(f) \leq \ell^2 + \ell$.*

Proof. If

Proof. 1. If A is a local Artin ring, then principal ideal ring (PIR) is equivalent to having Zariski tangent dimension less than or equal to 1.
2. $\mathcal{O}_k$ is isomorphic to a finite direct product of Dedekind domains.
3. $R_k = \mathbb{T}_k/\ell\mathbb{T}_k$.
The proof is by way of contradiction, so assume that ℓ doesn't divide the index $[\mathcal{O}_k : \mathbb{T}_k]$. Then, the map $: \mathcal{O}_k/\mathbb{T}_k \to \mathcal{O}_k/\mathbb{T}_k$ given by multiplication by ℓ is an isomorphism, and so $\mathcal{O}_k = \ell\mathcal{O}_k + \mathbb{T}_k$. Note that $\mathbb{T}_k$ is a subring and $\ell\mathcal{O}_k$ is an ideal in $\mathcal{O}_k$. We have,

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