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Uniqueness of a three-dimensional stochastic differential equation

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One motivation for studying higher-dimensional SDEs comes from the wave equation

$$\begin{aligned}\partial_t^2 u &= \Delta u,\\ u(0,x) &= u_0(x),\\ \partial_t u(0,x) &= u_1(x).\end{aligned}$$

In this equation, we have

$$\partial_t^2 u = \partial_x^2 u = \Delta u.$$

If we let

$$v = \partial_t u,$$

then we can rewrite the wave equation as the following system of equations:

$$\begin{aligned}\partial_t u &= v,\\ \partial_t v &= \Delta u.\end{aligned}$$

The original wave equation includes no noise. However, many physical systems are affected by noise. Hence, a modification of the wave equation which includes white noise is also studied:

$$\begin{aligned}\partial_t^2 u &= \Delta u + f(u)\dot{W},\\ u(0,x) &= u_0(x),\\ \partial_t u(0,x) &= u_1(x).\end{aligned}\tag{1}$$

Note $x \in \mathbb{R}$ and $\dot{W} = \dot{W}(t,x)$ is white noise.

One well-known point is that Lipschitz continuity is sufficient for the uniqueness of SDEs. Thus, many mathematicians have studied whether Hölder continuity can still ensure the uniqueness property of SDEs. Gomez, Lee, Mueller, Neuman, and Salins [Gomez et al. 2017] studied the uniqueness property of the following two-dimensional model of SDEs:

$$\begin{aligned}dX &= Y\,dt,\\ dY &= |X|^{\alpha}\,dB,\\ (X_0,Y_0) &= (x_0,y_0).\end{aligned}\tag{2}$$

The results focused on $f(x) = |x|^{\alpha}$ since it is a prototype of an equation with Hölder continuous coefficients.

It was proven in [Gomez et al. 2017] that if $> \frac{1}{2}$ and $.x_0; y_0$

the absolute values of its components,

$$|\vec{v}|_{\ell^\infty} = \max\{|v_i| : i = 1, 2, \dots, n\};$$

and

$$a \wedge b = \min(a, b);$$

$$a \vee b = \max(a, b).$$

If there is no such time, we let τ_n be infinity. Note that $\lim_{n \uparrow \infty}(\tau_n) = \infty$.

Now, for each fixed n, we will show uniqueness up to time τ_n in the system of equations

$$\begin{aligned} dX_t^{i,n} &= Y_t^{i,n}\,dt; \\ dY_t^{i,n} &= Z_t^{i,n}\,dt; \\ dZ_t^{i,n} &= |X_t^{i,n}|^{\gamma}\,\mathbf{1}_{[0,\tau_n]}(t)\,dB_t; \\ (X_0; Y_0; Z_0) &= (x_0; y_0; z_0). \end{aligned} \tag{4}$$

In other words, after the time τ_n, we have $dZ_t^{i,n} = 0$, which makes $Z_t^{i,n}$ become constant. Specifically, given $m, n \in N$, we need

$$(X_t^n; Y_t^n; Z_t^n) = (X_t^m; Y_t^m; Z_t^m)$$

for all $t \leq \tau_n \wedge \tau_m$.

Now, before continuing the proof of uniqueness, by the method of contradiction, we show that the times that X_t hits zero do not accumulate before the time τ_n, almost surely.

For each n, let A_n be the event on which the times that $X_t^{i,n} = 0$, $i = 1$ or $i = 2$, accumulate before τ_n, and assume $P(A_n) > 0$. Then, on A_n, suppose σ_n is an accumulation point of the times t at which $X_t^{i,n} = 0$; i.e, there exists a sequence of times $\sigma_{1,n} < \sigma_{2,n} < \cdots$ that converges to σ_n, and $X_{\sigma_{k,n}}^{i,n} = 0$. Hence, on A_n, $\lim_{k \to \infty} \sigma_{k,n} = \sigma_n$.

We have $X_t^{i,n}$ is almost surely continuous, and that $X_{\sigma_{k,n}}^{i,n} = 0$ on A_n, so

$$\lim_{\sigma_{k,n} \to \sigma_n} X_{\sigma_{k,n}}^{i,n} = X_{\sigma_n}^{i,n} = 0$$

on A_n.

Note that $dX_t^{i,n} = Y_t^{i,n}\,dt$, and $Y_t^{i,n}$ is almost surely continuous. So if $Y_{\sigma_n}^{i,n} \neq 0$ on A_n, then there exists a random interval $[\sigma_n(\omega) - \delta(\omega), \sigma_n(\omega)]$ of positive length for which $X_t^{i,n} \neq 0$ on $[\sigma_n(\omega) - \delta(\omega), \sigma_n(\omega)]$. This contradicts the hypothesis of $\sigma_{k,n}$ converging to σ_n.

If $Y_{\sigma_n}^{i,n} = 0$, there are two cases,

$$\sigma_n \geq \tau_n \quad \text{and} \quad \sigma_n < \tau_n.$$

If $\sigma_n \geq \tau_n$, then it means that the times at which $X_t^{i,n}$ hit zero do not accumulate before τ_n.

If $\tau_n < \sigma_n$, then with $X^{i,n}_{\tau_n} = 0$ and $Y^{i,n}_{\tau_n} = 0$, we have $|Z^{i,n}_{\tau_n}| > 2^{-n}$. Now suppose $Z^{i,n}_{\tau_n} > 2^{-n}$, as the case $Z^{i,n}_{\tau_n} < -2^{-n}$ is approached similarly due to symmetry.

If $Z^{i,n}_{\tau_n} > 2^{-n}$, since Z_t is almost surely continuous, there exists a time interval $[\tau_n(\omega) - \delta'(\omega), \tau_n(\omega)]$ on which $Z^{i,n}_t > 2^{-n}/2$. Thus for all $t \in [\tau_n(\omega) - \delta'(\omega), \tau_n(\omega)]$ we almost surely have

$$Y^{i,n}_t = Y^{i,n}_{\tau_n} - \int_t^{\tau_n} Z^{i,n}_s \, ds < -\frac{2^{-n}}{2}(\tau_n - t).$$

Hence, integrating over $X^{i,n}_t$ for $t \in [\tau_n - \delta', \tau_n]$, we have

$$X^{i,n}_t = X^{i,n}_{\tau_n} - \int_t^{\tau_n} Y^{i,n}_s \, ds = -\int_t^{\tau_n} Y^{i,n}_s \, ds > \int_t^{\tau_n} \frac{2^{-n}}{2}(\tau_n - s) \, ds$$

$$= \frac{2^{-n}}{2}\left(\tau_n s - \frac{s^2}{2}\right)\Big|_t^{\tau_n} = \frac{2^{-n}}{4}(\tau_n - t)(\tau_n - t)^2$$

Hence

$$X_t^{i,n} \le X_0^{i,n} + \int_0^t \frac{2^{-n}}{2}\, ds \le \frac{2^{-n}}{2} t. \tag{8}$$

Furthermore, based on (8) and $|X_t^{i,n}| \le 2^{-n}$ for all $t \in [0, t_{0,n}]$, it leads to $t_{0,n} \ge 2$, otherwise $X_t^{i,n} > 2^{-n}$, which means that $t > t_{0,n}$, a contradiction.

Note that

$$X_t^{i,n} = X_0 + \int_0^t Y_s^{i,n}\, ds;$$
$$Y_s^{i,n} = Y_0 + \int_0^s Z_k^{i,n}\, dk;$$
$$Z_k^{i,n} = Z_0 + \int_0^k |X_r^{i,n}|\, \mathbf{1}_{[0,\tau_n]}(t)\, dB_r.$$

Thus

$$X_t^{i,n} = X_0 + Y_0 t + \int_0^t \int_0^s \left(Z_0 + \int_0^k |X_r^{i,n}|\, \mathbf{1}_{[0,\tau_n]}(t)\, dB_r \right) dk\, ds$$
$$= X_0 + Y_0 t + \int_0^t \int_0^s Z_0\, dk\, ds + \int_0^t \int_0^s 0$$

$$
\begin{aligned}
&\le t^2 E \int_0^t \int_0^s \int_0^k \left(|X_r^{1,n}|^\gamma - |X_r^{2,n}|^\gamma\right)^2 dr\,dk\,ds \\
&\le t^2 E \int_0^t \int_0^t \int_0^t \left(|X_r^{1,n}|^\gamma - |X_r^{2,n}|^\gamma\right)^2 dr\,dk\,ds \\
&= t^4 E \int_0^t \left(|X_r^{1,n}|^\gamma - |X_r^{2,n}|^\gamma\right)^2 dr.
\end{aligned}
$$

Now we apply the mean value theorem for the function $f(x) = x^\gamma$, $0 < \gamma < 1$, and $a < b$:

$$b^\gamma - a^\gamma = \gamma c^{\gamma-1}(b-a) \le \gamma a^{\gamma-1}(b-a)$$

for $c \in (a, b)$. Then for $r \in [0, t_{0,n}]$, where t_0 is determined in (5), we apply (8):

$$\left| |X_r^{1,n}|^\gamma - |X_r^{2,n}|^\gamma \right| \le \gamma \left(\frac{2^{-n}}{2} r\right)^{\gamma-1} \left| |X_r^{1,n}| - |X_r^{2,n}| \right|. \tag{9}$$

Now let

$$D_t = E\left[\left(|X_t^{1,n}| - |X_t^{2,n}|\right)^2\right].$$

Since $t_{0,n} \le 2$, we have for all $t \in [0, t_{0,n}]$

Note that for all $t \in [0, t_{0,n}]$, we have $Y_t^{i,n} > 2^{-n}/2 > 0$. Hence $X_t^{i,n}$ is strictly increasing, which, leads to $X_{t_{0,n}}$ strictly positive. Therefore, by the strong Markov property, we have uniqueness until the time X next hits zero.

<u>Case II</u>: $Y_0 > 0$, $Z_0 <$ $<$ Hence

<u>Case III</u>: $Y_0 > 0$, $Z_0 > 2^{-n}$. Now we let T_n be the first time that either $Z_t^{1;n}$ or $Z_t^{2;n}$ hits the value $2^{-n}/2$. Since both $Z_t^{1;n}$ and $Z_t^{2;n}$ are continuous, we have $T_n > 0$ with probability 1. So we now prove uniqueness up to the time $t_{0;n} \wedge T_n$, where $t_{0;n}$ is defined in (5).

Then for all t in $[0, t_{0;n} \wedge T_n]$, we have

$$Z_t^{i;n} \geq \frac{2^{-n}}{2},$$

therefore

$$Y_t^{i;n} \geq Y_0 + \int_0^t \frac{2^{-n}}{2} ds \geq \frac{2^{-n}}{2} t \tag{11}$$

since $Y_0^{i;n} \geq 0$.

Based on (11), for $t \in [0, t_{0;n} \wedge T_n]$

$$X_t^{i;n} \geq X_0 + \int_0^t \frac{2^{-n}}{2} s \, ds \geq \frac{2^{-n}}{4} t^2$$

Using (12) and the mean value theorem, for $r \in [0, \tau_n \wedge T_n \wedge t_{0,n}]$, we have

$$\left| |\tilde{X}_r^{1,n}|^{\alpha} - |\tilde{X}_r^{2,n}|^{\alpha} \right| \le \alpha \left(\frac{2^{-n}}{4} r^2 \right)^{\alpha - 1} \left| |\tilde{X}_r^{1,n}| - |\tilde{X}_r^{2,n}| \right| .$$

Hence

$$E[(\tilde{X}_t^{1,n} - \tilde{X}_t^{2,n})^2] \le t^{4\alpha - 2} \left(\frac{2^{-n}}{4} \right)^{2(\alpha - 1)} E \int_0^t r^{4(\alpha - 1)} (|\tilde{X}_r^{1,n}| - |\tilde{X}_r^{2,n}|)^2 \, dr;$$

so if we let

$$D_t = E[(|\tilde{X}_t^{1,n}| - |\tilde{X}_t^{2,n}|)^2];$$

then

$$D_t \le E[(\tilde{X}_t^{1,n} - \tilde{X}_t^{2,n})^2] \le C_n \int_0^t r^{4\alpha - 4} D_r \, dr.$$

Again, applying Gronwall's lemma, with Dc with $(\tilde{X}_0^{i;n}; \tilde{Y}_0^{i;n}; \tilde{Z}_0^{i;n}) = (X_0; Y_0; Z_0)$ for $i = 1; 2$. Furthermore, using (13), $\tilde{X}_t^{i;n}$, $\tilde{Y}_t^{i;n}$, $\tilde{Z}_t^{i;n}$ can be defined for all time.