Estimates on constants related to Minkowski dimension

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Abstract

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In this paper we develop the theory of fractal dimension, introducing several de nitions of and concepts related to the Minkowski and Hausdor dimensions of a set. After reporting some results from [1], we provide some bounds on constants that in turn determine bounds on intersections of sets of di erent dimensions.
Issues related to the shortcomings of this approach are discussed, in particular the fact that all the theorems hold up to a factor of in the exponent, and how this introduces signi cant limitations to the scope of this paper.

1 Introduction

The study of fractal geometry is usually ascribed to Mandelbrot, who coined the term fractal in 1975 [5], although, as it's often the case in mathematics, the idea of fractals and fractal dimension actually emerged from the work of previous mathematicians. Some of the most famous names are Weierstrass, Cantor, Hausdor , Fatou, Julia.

In pop culture, the idea of fractals is associated to beautiful self-similar shapes such as the Mandelbrot set or the growth patterns of cauli owers. There are, however, notions of fractals that are not restricted to strictly self-similar set. These notions usually rely on some sort of "statistical similarity" or "scale invariance" of a set, and can be de ned rigorously, as we shall see.

It is important to note that these generalized versions of self-similarity are not pointless abstraction, but can be found everywhere in the world around us: the shape of the delta of a river, or the "jagged-ness" of the coast of an island [4], even in the behavior of prices in the stock market [1] can all be analyzed with these tools.

The rst notion of non-integer dimension was proposed by Hausdor in 1918 [3]. This de nition, known as Hausdor dimension, extends the usual notions of dimension to allow for non-integer values, and is still widely used. According to [7], the Hausdor dimension is considered more "robust", and is treated as a "standard".
As one would expect, the Hausdor dimension of "usual sets", such as lines, planes, or spheres, is exactly what one would expect. More generally, the Hausdor dimension of an n-dimensional smooth manifold exists and is n.

There are other de nitions of dimension that one can use to analyze sets. In this paper we are going to present two of them: the Minkowski dimension, and the discrete Hausdor dimension.
The former is a very useful tool in concrete cases, as it lends itself well to computations, but it loses some of the nice properties that the Hausdor dimension has.
As we shall see, this de nition tries to capture the fact that if a d-dimensional object is scaled by a factor of , its volume will scale roughly as:

$$V \quad {}^{d}$$

In particular, we will see that the numumumAs28()28(um)mlar,As29331(that)-34(scale)-29331(thatdiam3

The most straightforward way to elaborate on this is the Minkowski dimension, which will be the topic of the next section.

3 Minkowski dimension

This section closely follows Ch. 2 of [2].
One way of formalizing the notion of scaling described above that of Minkowski dimension.
There are various ways of defining this concept. Here we report two definitions, show that they coincide and explain why both are useful.
We will start with the heuristics.

Let us introduce a handy definition, that will recur throughout the paper;

Definition 3.1 (δ-cover) Given $E \subseteq \mathbb{R}^d$, we say that the (at most countable) collection of sets $\{U_i\}_i \subseteq \mathbb{R}^d$ form a δ-cover of E if $\mathrm{diam}(U_i) \leq \delta$ and $E \subseteq \bigcup_i U_i$

From this point and throughout this paper we will assume that $E \subseteq [0,1]^d$. This assumption doesn't really impact the main ideas presented here, but it does simplify some proofs.

Where $N(E)$ is de ned as above.

If we now take the lim sup (lim inf) of these two inequalities as $\to 0$ we see that the left and right sides both approach the upper (lower) Minkowski dimension.

While these two definitions yield the same dimensions, they have different applications. The first definition is well-suited for theoretical analysis, as it provides a minimal value without the need to construct it explicitly.
The box counting definition, on the other hand, is more useful for numerical

We can then define the s dimensional Minkowski content:

Definition 3.5 (Minkowski content) The upper and lower Minkowski contents of E are defined as:

$$\begin{aligned} \overline{M}^s(E) &= \limsup_{\epsilon \to 0} (2\epsilon)^{s-d} \mathcal{L}_d(E(\epsilon)) \\ \underline{M}^s(E) &= \liminf_{\epsilon \to 0} (2\epsilon)^{s-d} \mathcal{L}_d(E(\epsilon)) \end{aligned} \tag{10}$$

One can then define the upper and lower Minkowski dimensions using this concept:

$$\overline{\dim}_B(E) = \inf\{s : \overline{M}^s(E) = 0\} = \sup\{s : \overline{M}^s(E) > 0\}$$
$$\underline{\dim}_B(E) = \inf\{s : \underline{M}^s(E) = 0\} = \sup\{s : \underline{M}^s(E) > 0\}$$

a These definitions are equivalent to all the ones we've shown before, altough we will not show that here. We will instead prove some useful bounds.

Proposition 2 We have the following inequality

$$P_\epsilon(E)(\alpha_d)\epsilon^d \leq \mathcal{L}_d(E(\epsilon)) \leq N_\epsilon(E)(\alpha_d)(2\epsilon)^d \tag{11}$$

Proof: the fact that any ϵ-packing of E is contained in the ϵ-neighborhood of E proves the first inequality: the LHS is exactly the d-dimensional Lebesgue measure of the packing, so the result follows by the monotonicity of $\mathcal{L}_d$.

The second inequality can be proved by noting that if we replace the sets in a ϵ-cover with balls of radius $\epsilon/2$ that contain the original sets we still cover E. If the radius is then increased to 2ϵ we are guaranteed that $E(\epsilon)$ is also covered. The 2ϵ bound can be easily made sharper, but this will suffice for our purposes. If we call these balls with radius 2ϵ B_i, $1 \leq i \leq n$ we get:

$$E(\epsilon) \subseteq \bigcup_{i=1}^n B_i$$

By monotonicity and subadditivity of $\mathcal{L}_d$ we get:

$$\mathcal{L}_d(E(\epsilon)) \leq \sum_{i=1}^n \mathcal{L}_d(B_i) \leq N_\epsilon(E)(\alpha_d)(2\epsilon)^d$$

Thus proving the inequality.

3.2 Lipshitz functions

Here we briefly discuss the role that Lipshitz functions play in dimension theory. These functions are important as they preserve the Minkowski dimension of sets.

Definition 3.6 A function from $\mathbb{R}^n$ to $\mathbb{R}^m$ is said to be Lipshitz, or Lipshitz continuous, if there exists a positive real number L such that, for all $x, y \in \mathbb{R}^n$ we have:

$$|f(x) - f(y)| \leq L|$$

Definition 4.2 (s-dimensional Hausdorff measure) Let E be a subset of $\mathbb{R}^d$ and let $H^s_\delta(E)$ be defined as above. Then we defined the s-dimensional Hausdorff measure of E as:

$$H^s(E) := \lim_{\delta \to 0} H^s_\delta(E) \tag{14}$$

It can be shown that H^s is indeed a measure (in the measure-theoretic sense) and that it coincides with the Lebesgue measure (up to a constant factor depending on s) when s is an integer.

We will now see that, for every set E, there exists a unique value of s such that $H^s(E)$ is finite.
Let U_i be a δ-cover of E, and suppose $t > r \geq 0$.
Then we have:

$$\sum_i \operatorname{diam}(U_i)^t = \sum_i \operatorname{diam}(U_i)^{t-r} \operatorname{diam}(U_i)^r \leq \delta^{t-r} \sum_i \operatorname{diam}(U_i)^s$$

It can be shown that the inequality still holds when taking the infimum over δ-covers, to obtain:

$$H^t_\delta(E) \leq \delta^{t-r} H^r_\delta(E) \tag{15}$$

Note that (15) implies that if $r \neq t$ then $H^t(E)$ and $H^r(E)$ cannot be both finite and non-zero. More concretely, suppose that $H^r(E) < \infty$ and that $r < t$, then (15) must hold for any δ. By taking the limit as $\delta \to 0$, we see that the only way to satisfy (15) is if $H^r(E) = 0$.
Similarly, if $0 < H^t(E)$, the only way to satisfy (15) is if $H^r(E) = \infty$.
We summarize the above paragraph in the following definition and theorem.

Theorem 1 (Uniqueness of Hausdorff dimension) For any given $E \subseteq \mathbb{R}^d$, there exists at most one real number $s > 0$, such that $0 < H^s(E) < 0$.
In particular, if $0 < H^s(E) < 0$, and t and r are real numbers satisfying $0 < r < s < t$, we have $H^t(E) = 0$, $H^r(E) = \infty$,

Definition 4.3 (Hausdorff dimension) When such a value s exists such that $0 < H^s(E) < 0$ we say that the set E has Hausdorff dimension s.

Theorem 1 shouldn't come as a surprise. After all, the same is true with the usual Lebesgue measure: for example, if λ_d is the d-dimensional Lebesgue measure and Q_2 is the two dimensional unit cube (unit square), we have $\lambda_1(Q_2) = \infty$, $\lambda_2(Q_2) = 1$, and $\lambda_3(Q_2) = 0$.
This captures the idea that a unit square has "unit area", "zero volume", and "infinite length".

While Hausdorff dimension is a very powerful tool to analyze and characterize subsets of R=d, the following theorem tells us that we need a different tool to deal with countable sequences of finite sets.

Theorem 2 (Hausdorff dimension of countable sets) If $S \subset \mathbb{R}^d$ is countable, then it has Hausdorff dimension zero.

One idea that will allow us to analyze the dimension of countable sets is a modification of the Hausdorff dimension known as Discrete Hausdorff dimension. This will be the subject of the next section.

5 Discrete Hausdorff dimension

We now introduce a very important tool in the development of the idea of discrete Hausdorff dimension: the energy integral.

Definition 5.1 (Energy integral) Given a finite set $P_n \subset [0;1]^d$ with $|P_n| = n \in \mathbb{Z}^+$, we define the discrete r-energy of P_n as:

$$I_r(P_n) := n^{-2} \sum_{p \neq p'} |p - p'|^{-r} \tag{16}$$

This quantity is referred to as the "energy" of a finite point set, as it mimics the electric potential energy of a finite set of point particles with identical charges, in the case when $r = 2$.

Definition 5.2 (Time series) A time series is a collection P of sets $P_n \in [0;1]^d$ with $|P_n| = n$.
If all P_n are subsets of a set $E \subset [0;1]^d$ we say that P is a time series of E.

Definition 5.3 (Discrete Hausdorff dimension) Given a time series $P = \{P_n\}_{n \in \mathbb{Z}^+}$, we defined it's discrete Hausdorff dimension $\dim_{H_D}(P)$ as:

$$\dim_{H_D}(P) := \sup\{r \in [0;d] : \sup_n I_r(P_n) < \infty\} \tag{17}$$

Now that we have our basic ideas set up, we are almost ready to report some results from [1], which we will then build upon.

Before we introduce the results, let us take a small digression to discuss an issue in the definition of N_ϵ.

5.1 Freedom in choosing $N_\epsilon(E)$

When proving the equality of various definitions of Minkowski dimension we used the fact that we could multiply whatever definition of $N_\epsilon(E)$ we chose by a constant, say k, as the latter would be suppressed by a $\log(\epsilon^{-1})$ term:

$$\limsup_{\epsilon \to 0} \frac{\log(k N_\epsilon(E))}{\log(\epsilon^{-1})} = \limsup_{\epsilon \to 0} \frac{\log(k)}{\log(\epsilon^{-1})} + \frac{\log(N_\epsilon(E))}{\log(\epsilon^{-1})} = \limsup_{\epsilon \to 0} \frac{\log(N_\epsilon(E))}{\log(\epsilon^{-1})}$$

As the first term goes to zero in the limit.

There is no reason to require that k be a constant: it may as well be a function of (call it k( )), as long as it grows slow enough:

$$\limsup_{! \ 0} \frac{\log(k(\ ))}{\log(\ ^{1})} = 0 \qquad (18)$$

Remark 1 k( ) must go to zero or in nity slower than any polynomial.

Proof : 8 > 0 there must exist a > 0 such that 0 < < implies k( ) < C , where C is a constant. If we considered the lower Minkowski dimension we would get a similar inequality: k( ) > c .

This requirement is to be expected: if k( ) behaved like a polynomial, it would a ect the exponent in N C ^{s}.

We see that the property that gives the Minkowski dimension its exibility

The point of defining the discrete Hausdorff dimension is that it allows us to compute the notion of dimension of a set by simply sampling points. Theorem 4 of [1] provides a connection between the Minkowski measure of a set and the DHD of any of its discret subsets in the following sense:

Theorem 3 (Lower bound for $I_r(P_n)$ - Thm 4 of FRACTALS) Let P be a family of point sets contained in a subset $E \subseteq [0;1]^d$ of upper Minkowski dimension s.
Then, $\dim_{H_D} P = r \leq s$. If, instead, we have $r > s$, we have the following quantitative lower bound:

$$I_r(P_n) \geq \frac{s}{r-s} C_E^{-1-\frac{r}{s}} n^{\frac{r}{s}-1} - \frac{C_E^{-1} r}{r-s} + \frac{1}{n} \qquad (19)$$

Where $P_n \in P$ and C_E is as in (2).

Proof : see appendix.

A direct application of Thm 3. is that it constrains our ability to approximate a fractal set with smooth surfaces. This is explain concretely by Thm 8. from [1].

Theorem 4 (Intersection of sets with different dimensions) Let $P = \{P_n\}$ be a time series with $\dim_{H_D}(E) = s$ and let E be a subset of $[0;1]^d$ with $\overline{\dim}_B(E) = r > s$. Then, for every $\epsilon > 0$, there exists a constant C_ϵ such that:

$$|P_n \cap E| \leq C_\epsilon n^{\frac{2}{1+r/s}+\epsilon} \qquad (20)$$

In Thm 4. P is the set we are trying to approximate with E. What it means concretely is that if the dimension of P is greater than that of E, we will not be able to approximate it well.
For a concrete example example, suppose we wished to approximate a set $P \subseteq R^2$ with DHD greater than one by a smooth line (which has UMD equal to one). What Thm 4. tells us is that, no matter how well we approximate a finite subset of P, if we try to add more points most of them will lie outside of the approximating curve.

Note the presence of ϵ at the exponent, and of the multiplicative constant C_ϵ.
Proof : This proof follows that of [1], while accounting for constants that in that paper are lumped in to C_ϵ. In this case, C_ϵ is comes from the constant C_E in Lemma 1, which then ripples through the proofs.

Let $P'_m = P_n \cap E$, $m = |P'_m|$.
Then we have:

$$I_r(P'_m) = m^{-2} \sum_{\substack{p,p' \in P'_m \\ p \neq p'}}$$

Where C_s is the constant for which $I_s(P_n) \quad C_s$. This exists by hypothesis, as $r > s$.
We now apply Thm 3:

$$I_r(P^0_m) \quad C_m \frac{s}{r \quad s} \; C_E{}^{1 \; \frac{r}{s}} m^{\frac{r}{s} \; 1} \tag{22}$$

Where C_m is a constant that guarantees that:

$$\frac{s}{r \quad s} \; C_E{}^{1 \; \frac{r}{s}} m^{\frac{r}{s} \; 1} \qquad \frac{C_E{}^{1} \; r}{r \quad s} + \frac{1}{m}$$

Note that C_m can be taken to be arbitrarily close to 1 as n goes to in nity.

Combining this with (21) we obtain:

$$C_s m \;\; {}^2 n^2 \quad C_m \frac{s}{r \quad s} \; C_E{}^{1 \; \frac{r}{s}} m^{\frac{r}{s} \; 1}$$

We can use this to nd a bound for m:

$$m \qquad \left(\frac{C_S C_E^{\frac{r}{s}}}{s C_m} (r \quad s) \right)^{\frac{1}{\frac{r}{s}+1}} n^{\frac{2}{\frac{r}{s}+1}}$$

However, recall that C_E depends on , which we have no control over, so we will have to say

$$m \;.\; \qquad \left(\frac{C_S C_E^{\frac{r}{s}}}{s C_m} (r \quad s) \right)^{\frac{1}{\frac{r}{s}+1}} n^{\frac{2}{\frac{r}{s}+1}} \tag{23}$$

7 Results

In this section we report the main results of the paper. These results are estimates on the value of the constant C_E on sets satisfying some given properties. In particular, the bounds are given by the Hausdor measure, and the upper and lower Minkowski contents of the set E.

As explained before, bounding this constant allows one to make the statement of theorem 4 more precise.
Notation: in what follows we will often omit the reference to the set E. For example, we will write N in lieu of $N\;(E)$

We are now ready to state our results. We will start with a bound in the case the Hausdor measure of E can be computed.

Proposition 4 (Lower bound for C_E) Suppose $\dim_H(E) = \overline{\dim}_B(E) = s$. Then, For any $\;> 0$ There exist $\;> 0$ such that $8\;; 0 < \quad <$ we have $N \; > (V \quad)\;\;{}^{s}$.
Where we de ned $V := H^s(E) \quad H^s(E)$.

Proof :
$H^s(E) = \lim_{\to 0} H^s(E)$ and $H^s(E)$ is non decreasing as decreases so, given, there exists such that $\forall$, $0 < \; <$ we have $V \quad H^s(E) < \;)$.

$$H^s(E) > V \tag{24}$$

By definition, we have

$$H^s(E) \quad \sum_i |U_i|^s \text{ for any } \{U\}_i \text{ that is a}$$

Once again, recognize the LHS to be as in the de nition of A( ). Apply the least upper bound proposition: for su ciently small:

$$(\underline{M}^s \quad) \quad N \quad ^s \; 2^s \; (\; d)$$

Rearranging:

$$\begin{aligned} N \quad & ^s \; (\underline{M}^s \quad) \; \frac{1}{2^s (\; d)} \\ N \;\; & \&_{s;d} \quad ^s \; (\underline{M}^s \quad) \end{aligned} \tag{26}$$

We also provide a converse theorem:

Theorem 6 For small enough we have

$$N \; . \,_d \quad ^s \; (\overline{M}^s(S) + \quad)$$

Where $\overline{M}^s(S)$ is the s-dimensional upper Minkowski content of S, and where the constant implicit in $._d$ is equal to $\frac{2^d}{(\; d)}$.

Proof :
We restate equations (9) and (11) for reference:

$$N_2 \; (S) \quad P \; (S) \tag{27}$$

$$P \; (S) \quad (\; d) \quad ^d \quad _d(S(\;)) \tag{28}$$

Start from (11) with replaced by $\frac{}{2}$:

$$P_{\frac{}{2}}(S) \quad (\; d) \quad \frac{}{2}^{\,d} \quad _d \;\; S \; \frac{}{2}$$

Use (9) and obtain:

$$N \; (S) \quad (\; d) \quad \frac{}{2}^{\,d} \quad _d \;\; S \; \frac{}{2}$$

Multiply both sides by $\frac{}{2}^{\;s \;\; d} = \;\; ^{s \;\; d}$:

$$N \; (S) \quad (\; d)^{\; d} \quad ^{s \;\; d} \; 2^{\; d} \quad _d \;\; S \; \frac{}{2} \quad ^{s \;\; d}$$

Simplifying, and recognizing that the RHS is just like in the de nition of Minkowski content :

$$2^{\; d} \quad (\; d) N \; (S) \quad ^s \quad \overline{M}^s(S) +$$

Finally, solving for $N \; (S)$:

$$\begin{aligned} N \; (S) \quad & ^s \quad \overline{M}^s(S) + \quad \frac{2^d}{(\; d)} \\ N \; (S) \; . \,_d \quad & ^s \quad \overline{M}^s(S) + \end{aligned} \tag{29}$$

8 Conclusions

In this paper we reviewed the theory of Minkowski and Hausdor dimension, providing several di erent de nitions and showing their equality and di erent use cases. After summarizing some results from [1], we showed how one can nd some bounds for the constants involved in the theorems using the Minkowski content of the set under examination. The procedure used has a major shortcoming, in that the bounds only hold up to a factor of in the exponent. In order to ensure control over this factor, one has to introduce another multiplicative constant, depending on . This then leaves the overall constant undetermined, Now one has to notice that $\sum_i |E_i \cap P_n| = |P_n| = n$ to conclude:

$$|\{(p, p') : |p - p'| \le \epsilon\}| \ge C_E^{-1} \epsilon^{s} n^2$$

Proof of Thm. 3.
Recall the form of the energy integral:

$$I_r(P_n) = n^{-2} \sum_{p \neq p'} |p - p'|^{-r} \tag{30}$$

We want to use what we know about the upper Minkowski dimension of E to bound this expression. The only tools we have that relates the UMD of a set to its time series is Lemma 2, so we are going to re-express (30) in a form that will allow us to apply Lemma 2.

Start from noticing that:

$$|p - p'|^{-r} = r \int_0^\infty 1_{[0,\infty)}(\epsilon - |p - p'|)\, \epsilon^{-r-1} d\epsilon$$

Hence:

$$\begin{aligned} I_r(P_n) &= n^{-2} \sum_{p \neq p'} |p - p'|^{-r} \\ &= r n^{-2} \sum_{p \neq p'} \int_0^\infty 1_{[0,\infty)}(\epsilon - |p - p'|)\, \epsilon^{-r-1} d\epsilon \end{aligned} \tag{31}$$

The sum above is finite, so we can swap the sum and the integral:

$$I_r(P_n) = r n^{-2} \int_0^\infty \left(\sum_{p \neq p'} 1_{[0,\infty)}(\epsilon - |p - p'|) \right) \epsilon^{-r-1} d\epsilon \tag{32}$$

Note that the sum in parenthesis just counts the number of point pairs that are less than ϵ apart, excluding the n pairs with $p = p'$:

$$I_r(P_n) = r n^{-2} \int_0^\infty (|\{(p, p') : |p - p'| \le \epsilon\}| - n)\, \epsilon^{-r-1} d\epsilon \tag{33}$$

Note that the expression above is exactly what we have in Lemma 2, which we are now going to apply: if $\epsilon \le C_E^{-1} n^{-1/s}$ we are only guaranteed the existence of the n pairs $p = p'$. and thus:

$$|\{(p, p') : |p - p'| \le \epsilon\}| - n \ge 0$$

If, instead, $\epsilon \ge C_E^{-1} n^{-1/s}$ Lemma 2 tells us more:

$$|\{(p, p') : |p - p'| \le \epsilon\}| - n \ge C_E^{-1} \epsilon^{s} n^{2}$$

Applying (34) to (33) we get:

$$
\begin{aligned}
I_r(P_n) &\geq rn^{-2}\int_{(C_E^{-1}n)^{-1/s}}^{1} \left(C_E^{-1}\lambda^{s}n^{2} - n\right)\lambda^{-r-1}\,d\lambda \\
&= rC_E^{-1}\int_{(C_E^{-1}n)^{-1/s}}^{1} \lambda^{s-r-1}\,d\lambda \\
&\quad - rn^{-1}\int_{(C_E^{-1}n)^{-1/s}}^{1} \lambda^{-r-1}\,d\lambda
\end{aligned}
\qquad (35)
$$

These integrals are finite, because we are assuming that $r > s$, and can be evaluated to:

$$
= \frac{r}{s-r}\, C_E^{-1-\frac{r}{s}}\, n^{\frac{r}{s}-1} - \frac{rC_E^{-1}}{r-s} + \frac{1}{n} \qquad (36)
$$