2 The Difference of Two Series

We will begin by laying down the foundation of this paper; that we can force the difference of two series defined by functions that monotonically decrease to zero to converge by carefully choosing the "density" of the two series with respect to each other. Density is put in quotations as what we will be using is slightly more nuanced than a ratio but it amounts to a very similar thing; it will be performed using $p(x;y)$ whereas normal density would just be $k\ x$. The y variable for p will be used later on, but not in this section. Throughout this paper we will be using the relevant definitions and basic theorems from Apostol's book on mathematical Analysis [3].

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ where $\mathbb{R}$ is understood to be $\{x | x >= 0\}$. We require that f, g have the following properties:

- f, g are monotone decreasing.
- $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$
- $\int_0^x f(t)dt$ and $\int_0^x g(t)dt$ exist and are finite $\forall x \geq 0$
- $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} g(n) = \infty$

We will define $F(x) = \int_0^x f(t)dt$ and $G(x) = \int_0^x g(t)dt$.

Remark. $F; G$ are strictly increasing and $F(\mathbb{R}_+) = G(\mathbb{R}_+) = \mathbb{R}_+$, so they both have inverse functions defined on $\mathbb{R}_+$.

Remark. By $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} g(n) = \infty$, we have

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} G(x) = \infty$$

Let us define $S \subset \mathbb{R}_+ \times \mathbb{R}$ as $S = \{(x;y) | F(x)+y \geq 0\}$. Let $p(x;y) : S \to \mathbb{R}_+$ be $p(x;y) = G^{-1}(F(x) + y)$.

Lemma 2.1. $p(x;y)$ is well defined, and p is monotone increasing with respect to both x and y, and $\lim_{x\to\infty} p(x;y) = \infty \ \forall y \in \mathbb{R}$.

Proof. Note by $(x;y) \in S$ $F(x) + y \geq 0 \Rightarrow F(x) + y \in \mathbb{R}_+$. Thus, $p(x;y) = G^{-1}(F(x) + y)$ is well defined. Note F and G are differentiable by the Fundamental Theorem of Calculus. Furthermore, clearly by $\lim_{x\to\infty} f(x) = 0$, $\sum_{n=1}^{\infty} f(n) = \infty$, and f monotone decreasing, we can conclude

$$f(x) > 0 \ \forall x \in \mathbb{R}_+$$

Similarly for g. Then we find that $F'(x) > 0$ and $G'(x) > 0$. Then we may conclude that $F^{-1}(x)$ is differentiable, and that $(F^{-1})'(x) > 0$. Similarly, $G^{-1}(x)$ is differentiable, and $(G^{-1})'(x) > 0$.

Then we may conclude that $p(x;y)$ is differentiable with respect to x and y, and by the chain rule

$$\frac{\partial p}{\partial x}(x;y) = (G^{-1}(F(x) + y))' = G^{-1\prime}(F(x) + y) \cdot F'(x) > 0$$

Thus p(x,y) is strictly increasing with respect to x.

Note that $lim_{x!1} \ G(x) = 1$ and G strictly increasing means that $lim_{x!1} \ G^{\ 1}(x) = 1$. By $lim_{x!1} \ F(x) + y = 1$ we may conclude that $\lim_{x!1} \ p(x;y) = 1$. □

Theorem 2.2. $\lim_{x!1} \quad P^{\lfloor x \rfloor}_{n=1} f(n) \quad P^{\lfloor p(x;0) \rfloor}_{n=1} g(n)$ exists and is nite.

Proof. We will show that the series is Cauchey by using the fact that the series di er from their respective integral by no more than the rst term of the series.

Let $h(x) = \ P^{\lfloor x \rfloor}_{n=1} f(n) \quad P^{\lfloor p(x;0) \rfloor}_{n=1} g(n)$ where $h : R_+ \ ! \ R$. Choose $m >$

We will now work on bounding it from below. Note that

$$h(m) \quad h(k) \quad \int_{k+1}^{m \quad 1} f(t)dt\ )$$

Theorem 3.2. $d(y) = d(0) - y$

Proof. This proof will largely hinge upon the fact that we have defined $p(x;y) = G^{-1}(F(x)+y)$, and so $F(x) - G(p(x;y)) = F(x) - G(G^{-1}(F(x)+y)) = F(x) - F(x) - y = -y$. Everything else amounts to managing the error terms for estimating series with integrals. If $y > 0$, Note

$$d(y) - d(0) = \lim_{x\to\infty} \sum_{n=1}^{\lfloor x\rfloor} f(n) - \sum_{n=1}^{\lfloor p(x;y)\rfloor} g(n) - \lim_{x\to\infty} \sum_{n=1}^{\lfloor x\rfloor} f(n) + \sum_{n=1}^{\lfloor p(x;0)\rfloor} g(n)$$

If we simplify, we may use Lemma 1.1 and Lemma 3.1 to conclude

$$d(y) - d(0) = -\lim_{x\to\infty} \sum_{\lfloor 1+p(x;0)\rfloor}^{\lfloor p(x;y)\rfloor} g(n)$$

Then we apply $g(n)$ monotone decreasing to change into integrals and see that

$$d(y) - d(0) \geq -\lim_{x\to\infty} \int_{p(x;0)+1}^{p(x;y)-1} g(t)dt$$

Once again by $g(n)$ monotone decreasing we may add back on the edges, canceling them out with the constant $g(p(x;0))$, and convert the main portion of the integral into

$$d(y) - d(0) \geq -\lim_{x\to\infty} G(p(x;y)) - G(p(x;0)) + 2g(p(x;0))$$

But by using the definition of $p(x)$, we have

$$d(y) - d(0) \geq -\lim_{x\to\infty} G(G^{-1}(F(x)+y)) - G(G^{-1}(F(x))) + 2g(p(x;0))$$

Then we find that

$$d(y) - d(0) \geq -y - \lim_{x\to\infty} 2g(p(x;0)) = -y$$

Now we will bound by the other side and say

$$d(y) - d(0) \leq -\lim_{x\to\infty} \int_{p(x;0)-1}^{p(x;y)+1} g(t)dt$$

so then using the previous steps, we have

$$d(y) - d(0) \leq -\lim_{x\to\infty} G(G^{-1}(F(x)+y)) - G(G^{-1}(F(x))) - 2g(p(x;0) - 1) = -y$$

Then we have $-y \leq d(y) - d(0) \leq -y$, so $d(y) = d(0) - y$.

If $y = 0$, we immediately see that $d(y) = d(0) = d(0) - 0$.

If $y > 0$, we have that

$$d(y) - d(0) = \lim_{x\to\infty} \sum_{\lfloor p(x;y)+1\rfloor}^{\lfloor p(x;0)\rfloor} g(n)$$

We can then see, applying the previously used logic, that

$$d(y) - d(0) \leq \lim_{x\to\infty} \int_{p(x;y)-1}^{p(x;0)+1}$$

$$d(y) - d(0) \leq \lim_{x\to\infty} G(p(x;0)) - G(p(x;y)) + 2g(p(x;y))$$

$$d(y) - d(0) \leq y$$

Additionally, bounding in the other direction, we have

$$d(y) - d(0) \geq \lim_{x\to\infty} \int_{p(x;y)+1}^{p(x;0)-1}$$

$$d(y) - d(0) \geq \lim_{x\to\infty} G(p(x;0)) - G(p(x;y)) - 2g(p(x;y))$$

$$d(y) - d(0) \geq -y$$

Then we once again have that $-y \leq d(y) - d(0) \leq y$, and so we are done with all three cases. □

Because $d(c) = b - c$ for some b, if we can calculate $d(c)$ for some value of c we are instantly able to calculate it for all values of c. If it is infeasible to calculate $d(c)$ precisely for any value, it becomes important to consider how many terms of the series one would have to calculate to obtain an accurate estimate. The following lemmas describe the rate at which the series converges do $d(c)$. I by no means claim that these are the best terms one can obtain; in fact I suspect that with use of better estimation tools such as Abel's identity as written in Apostol's Introduction to Analytic Number Theory [4] or some other similar estimate, better terms might be able to be obtained. Ultimately though this resolves down to that the series converges at least as quickly as the individual terms converge to zero.

Lemma 3.3. $d(C) - (\sum_{n=1}^{\lfloor x\rfloor} f(n) - \sum_{n=1}^{\lfloor p(x;C)\rfloor} g(n)) \leq (f(k-1) - f(\lfloor k\rfloor)) + g(\lfloor p(k;C)\rfloor)$

Proof. Let $h(x) = \sum_{n=1}^{\lfloor x\rfloor} f(n) - \sum_{n=1}^{\lfloor p(x;C)\rfloor} g(n)$ where $h : \mathbb{R}_+ \to \mathbb{R}$. Let $m > k > 0$ so $p(m;C) > p(k;C)$. Note $h(\infty) = d(C)$

Then note

$$h(\infty) - h(k) = \lim_{m\to\infty} \sum_{n=\lfloor k\rfloor}^{m} f(n) - \sum_{n=\lfloor p(k;C)\rfloor}^{\lfloor p(m;C)\rfloor} g(n) - f(\lfloor k\rfloor) + g(\lfloor p(k;C)\rfloor)$$

equation we get $\lim_{x \to \infty} h(x) - G^{-1}(F(x)+C) = \lim_{x \to \infty} h(x) - p(x, C) = 0$. Consider

$$\sum_{n=1}^{\infty} a_n - d(C) = 0$$

is equivalent to

$$\lim_{x \to \infty} \sum_{n=1}^{\lfloor h(x) \rfloor} g(n) - \sum_{n=1}^{\lfloor p(x,C) \rfloor} g(n) = 0$$

Note that $g(n)$ is monotone decreasing, and that

$$\exists M \, s.t. \forall x > M, |h(x) - p(x,C)| < 2$$

Then note that $|\lfloor h(x) \rfloor - \lfloor p(x,C) \rfloor| \leq 2$ Then $\forall x > M$ we have

$$\left| \sum_{n=1}^{\lfloor h(x) \rfloor} g(n) - \sum_{n=1}^{\lfloor p(x,C) \rfloor} g(n) \right| \leq 2g(\min(\lfloor h(M) \rfloor, \lfloor p(M,C) \rfloor))$$

Note that by $h(x) \to \infty$, $p(x,C) \to \infty$, and $g(x) \to 0$, we have that

$$\lim_{M \to \infty} 2g(\min(\lfloor h(M) \rfloor, \lfloor p(M,C) \rfloor)) = 0$$

Then we have that $g(n)(n)$ w ni -291.48case0 Td [Td [4323(=)-323(0.

$$\lim_{x \to \infty} \sum_{n=1}^{\lfloor h(x) \rfloor} g(n) \quad \lfloor p(x,C) \rfloor$$

To illustrate the use of these theorems and corollaries, we will recompute the fact that if c

converges for some y. Note that for the same y, we have the convergent series

$$\lim_{x\to\infty} \sum_{n=1}^{\lfloor x \rfloor} f(n) - \sum_{n=1}^{\lfloor p(x;y) \rfloor} g(n)$$

By the difference of limits is the limit of differences, we must have that

$$\lim_{x\to\infty} \sum_{n=1}^{\lfloor p(x;y) \rfloor} g(n) - \sum_{n=1}^{\lfloor p(x;y) \rfloor} a_n$$

converges. Note that $p(x;y)$ is continuous, strictly increasing with respect to x, and that $\lim_{x\to\infty} p(x;y) = \infty$. Then we may replace $m = p(x;y)$ and note that the following series converges, and thus we are done.

$$\lim_{x\to\infty} \sum_{n=1}^{\lfloor x \rfloor} g(n) - \sum_{n=1}^{\lfloor x \rfloor} a_n$$

We will proceed in the reverse direction by assuming that the above series

[1] Bernhard Riemann, *Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe,* Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Math. Klasse. (1866-67), 87-132.

[2] Augustin Louis Cauchy, *Summary of Lessons given at the Royal Polytechnic School on infinitesimal calculus*, Complete Works, Series 2, (1827)

[3] Tom M. Apostol. *Mathematical Analysis.* Addison-Wesley Publishing Company, Reading Massachusetts, 1981.

[4] Tom M. Apostol. *Introduction to Analytic Number Theory*. Springer-Verlag, New York, 1976.