Dionel Jaime

Method of Stationary Phase and Applications

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1 Introduction

In short, the method of Stationary Phase concerns itself with obtaining bounds for oscillatory integrals of the following form

$$I(f, R) = \int_{\mathbb{R}^n} e^{iRf(x)}\, \psi(x)\, dx$$

where $\psi(x)$ is a bump function.

The method of stationary phase is of great importance in the field of Harmonic Analysis. One can find applications of this to other areas in Harmonic Analysis, Partial Differential Equations, Geometric Measure Theory, and Geometric Combinatorics to name a few. Different bounds are obtained depending on the conditions imposed on f

Such an estimate is begging for a more general result. The first of these is the Vander-Corput lemma.

Lemma 2.1. *(Vander Corput) Suppose that* $f \in C^1(a,b)$, $|f'(x)| \geq 1$, *and* $f'(x)$ *is monotonic. Then*

$$|I(f,R)| = \left| \int_a^b e^{iRf(x)}\, dx \right| \leq \frac{2}{R}$$

Proof. We make use of the following trick

$$|I(f,R)| = \left| \int_a^b \frac{1}{iRf'(x)} \frac{d}{dx} \left(e^{iRf(x)} \right) dx \right| \tag{1}$$

$$= \left| \left(\frac{e^{iRf(b)}}{iRf'(b)} - \frac{e^{iRf(a)}}{iRf'(a)} \right) - \int_a^b e^{iRf(x)} \frac{d}{dx} \left(\frac{1}{iRf'(x)} \right) dx \right| \tag{2}$$

$$\leq \frac{1}{R} \left(\frac{1}{f'(b)} + \frac{1}{f'(a)} \right) + \left| \int_a^b e^{iRf(x)} \frac{d}{dx} \left(\frac{1}{iRf'(x)} \right) dx \right| \tag{3}$$

To deal with the last integral, we make use of the monotonicity of f' and the fundamental theorem of calculus

$$\left| \int_a^b e^{iRf(x)} \frac{d}{dx} \left(\frac{1}{iRf'(x)} \right) dx \right| \leq \int_a^b \left| e^{iRf(x)} \frac{d}{dx} \left(\frac{1}{iRf'(x)} \right) \right| dx \tag{4}$$

$$= \left| \int_a^b \frac{d}{dx} \left(\frac{1}{iRf'(x)} \right) dx \right| \tag{5}$$

$$= \left| \left(\frac{1}{iRf'(b)} - \frac{1}{iRf'(a)} \right) \right| \tag{6}$$

$$= \frac{1}{R} \left| \frac{1}{f'(b)} - \frac{1}{f'(a)} \right| \tag{7}$$

Putting these two estimate together, we obtain that

$$|I(f,R)| \leq \frac{1}{R} \left(\frac{1}{f'(b)} + \frac{1}{f'(a)} + \left| \frac{1}{} \right. \right.$$

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□

Three observations are important in this proof. First, notice that even though we only assumed that $f \in C^1(a,b)$, we are still allowed to talk about

$$\frac{d}{dx}\left(\frac{1}{iRf'(x)}\right)$$

because the monotonicity of $f'(x)$ guarantees that this derivative exists almost everywhere. Hence, we're considering that integral as a Lebesgue integral.

Secondly, the condition that $f'(x) \ge 1$ is far stronger than necessary. We could just suppose that $f'(x) \ge \lambda$ where $\lambda > 0$ and make the appropiate changes to have a trivially more general result.

Lastly, the computation that preceded the theorem shows that the constant 2 in the bound of $\frac{2}{R}$ is the best that we can do in terms of constants.

This result can be extended by considering derivatives of higher powers. Before stating this, result, we should consider some motivation. Previously, our motivation for Vander-Corput was the estimate

$$\left|\int_a^b e^{iRx}\,dx\right| \le \frac{2}{R}$$

Naturally, we'd like to look at integrals of the form

$$I(x^n,R) = \int_a^b e^{iRx^n}\,dx$$

In order to gain some intuition on what results we should expect, we recall the classic Fresnel Integral. I won't prove this result as I will prove a more general result in a moment; however, the Fresnel integral tell us that

$$\int_0^\infty e^{iRx^2}\,dx = \frac{1}{R^{1/2}}\int_0^\infty e^{ix^2}\,dx = \frac{1+i}{R^{1/2}}\sqrt{\frac{\pi}{8}}$$

By similar methods, we can also deduce that

$$\int_0^\infty e^{iRx^n}\,dx = C$$

where $C_n = \int_0^\infty e^{ix}$

Applying the induction hypothesis, we obtain

$$\left| \int_{x_0+\delta}^{b} e^{iRf(x)}\, dx \right| = \left| \int_{x_0+\delta}^{b} e^{iR\delta \frac{f(x)}{\delta}}\, dx \right| \leq C_{m-1}(R\delta)^{-1/(m-1)}$$

We can apply the exact same argument to $-f(x)$ in the first integral and obtain

$$\left| \int_{a}^{x_0-\delta} e^{iRf(x)}\, dx \right| \leq C_{m-1}(R\delta)^{-1/(m-1)}$$

Putting this together, we obtain that

$$|I(f,R)| \leq 2C_{m-1}(R\delta)^{\frac{-1}{m-1}} + 2\delta$$

If we choose $\delta = R^{-1/m}$, we have

$$|I(f,R)| \leq 2C_{m-1}(RR^{-1/m})^{\frac{-1}{m-1}} + 2R^{-1/m}$$

and since

$$\frac{-1}{m-1} + \left(\frac{-1}{m}\right)\frac{-1}{m-1} = \frac{-m}{m(m-1)} + \frac{1}{m(m-1)} \tag{8}$$

$$= \frac{1-m}{m(m-1)} \tag{9}$$

$$= \frac{-1}{m} \tag{10}$$

we finally have that

$$|I(f,R)| \leq (2C_{m-1} + 1)R^{-\frac{1}{m}}$$

where if $C_m = 2C_{m-1} + 1$, then our proof is complete. □

It may be worth nothing that we can play around with the constants by choosing δ in the above proof to be diﬀerent.

It is natural to wonder whether we can keep extending this result. The motivation we used suggests the consideration of fractional powers of x.

$$I(x^\alpha, R) = \int_a^b e^{iRx^\alpha}\, dx$$

where $\alpha > 1$.

To show that

Proof. Consider the quarter annulus with inner radius ϵ and outer radius r, denoted, $\lambda_{\epsilon,r}$. Let $\alpha \in (0,1)$. By Cauchy's Theorem we have that for all $r, \epsilon > 0$

$$\int_{\lambda_{\epsilon,r}} z^{\alpha-1}e^{iz} = 0$$

We can break up the contour we're integrating over into four parts yielding the following

$$0 = \int_{\epsilon}^{r} x^{a-1}e^{ix}\,dx + \int_0^{\pi/2} (re^{it})^{\alpha-1}e^{i(re^{it})}ire^{it}$$
$$- \int_{\epsilon}^{r} (ix)^{a-1}ie^{-x}\,dx - \int_0^{\pi/2} (\epsilon e^{it})^{\alpha-1}e^{i(\epsilon e^{it})}i\epsilon e^{it}$$

We will get the second integral and the fourth integral out of the way by using some basic integral estimates and basic results in measure theory. We proceed as follows

$$\left|\int_0^{\pi/2} i(\epsilon e^{it})^{\alpha}e^{i\epsilon e^{it}}\,dt\right| \le \int_0^{\pi/2} \left|i(\epsilon e^{it})^{\alpha}e^{i\epsilon e^{it}}\right|\,dt \tag{14}$$
$$= \int_0^{\pi/2} \epsilon^{\alpha}\left|e^{i\epsilon(cos(t)+isin(t))}\right|\,dt \tag{15}$$
$$= \int_0^{\pi/2} \epsilon^{\alpha}e^{-\epsilon sin(t)}\,dt \tag{16}$$

On this domain, our integrand is a family of functions indexed by ϵ which is uniformly bounded by $\epsilon^{\alpha}e$ on a set of finite measure and approaches 0 point-wise as $\epsilon \to 0$. Therefore, we may apply the dominated convergence theorem and obtain

$$\lim_{\epsilon\to 0}\int_0^{\pi/2} \epsilon^{\alpha}e^{-\epsilon sin(t)}\,dt = \int_0^{\pi/2} \lim_{\epsilon\to 0} \epsilon^{\alpha}e^{-\epsilon sin(t)}\,dt = 0$$

Similarly we have that

$$\left|\int_0^{\pi/2} (re^{it})^{\alpha-1}e^{i(re^{it})}ire^{it}\,dt\right| \le \int_0^{\pi/2} re^{-rsin(t)}\,dt$$

Recall that removing a point from the domain of integration, $[0,\pi/2]$ does not change the value of the integral. Therefore we may view the last integral as being over $(0,\pi/2]$. The purpose of doing this is because on

Theorem 3.1. *Let* T

$$\hat{\psi}(\xi)\; e^{\pi i R^{-1} \quad T^{-1}\xi},$$

We need to demonstrate the following claim: Let $\varphi_{i=1}^{M}$ be real valued smooth functions and assume that $\varphi(p) = 0$ and that $\nabla\varphi(p) = 0$. Lastly, let

$$\Phi = \prod \varphi_i$$

Then all partial derivatives of Φ of order less than $2M$ also vanish at p.

Proof. By the product rule any partial $D^{\alpha}\Phi$ is a linear combination of terms of the form

$$\prod_{i=1}^{M} D^{\beta_i}\varphi_i$$

with $\sum \beta_i = \alpha$. If $|\alpha| < 2M$, then some β_i must be less than 2, so by hypothesis all such terms vanish at p.

To prove our theorem, we need only differentiate $I(R)$ under the integral sign in order to obtain

$$\frac{d^k}{dR^k}(e^{-\pi i R(}$$

and all that is required of the remaining maps is that they map onto sets whose closures do not contain $\{\pm e_n\}$

Let φ_k

4 References

1. Thomas M Wolff. Lectures in Harmonic Analysis(Revised). March 2002

2. Pertti Matilla. Fourier Analysis and Hausdorff Dimension. Cambridge University Press 2015