SUGAWARA'S CONSTRUCTION OF VIRASORO ALGEBRA FOR $c = 1$, $h = 0$

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1. Introduction

Group theory provides a useful way of mathematically studying the symmetries of physical systems. Most common are finite dimensional Lie groups such as SU(n),U(n), and SO(n) which contribute to the framework of rotations. Additionally, finite groups have been used extensively in crystallography (Sternberg [1]). These examples maintain a strong focus on the finite dimensional symmetry and have many applications to quantum mechanics. However, to pass into quantum field theory, where it is important to consider systems with an infinite number of degrees of freedom, algebras such as the Virasoro and Kac-Moody algebras are necessary for infinite dimensional symmetries.

The Virasoro algebra arises naturally in various applications of physics. In a mathematical formulation, the Virasoro algebra can be viewed as coming from the algebra of the conformal group in one or two dimensions. Two-dimensional conformally invariant structures are present in many areas of physics, including 2-dimensional statistical systems of spins on lattices and in finding a mathematically consistent theory of particle interactions (Goddard and Olive [2]). Furthermore, the Virasoro algebra becomes useful in analyzing mass-less fermionic and bosonic field theories.

Due to its presence in numerous areas of physics, it is important to be able to construct representations for the Virasoro algebra. The Virasoro algebra is often associated with observable quantities. In particular, the algebra determines the mass spectrum in string theory, and also energy spectra in 2-dimensional quantum field theories made up of two copies of the algebra (Goddard and Olive [2]). In this sense, it is important for the representation of the Virasoro algebra to be highest weight, so that the energy spectrum is positive (or at least bounded below).

In addition to this requirement, it is necessary that the representations be unitary. In quantum mechanics,

Proof.
( ) Suppose there exists a linear map $f : \mathfrak{g} \to \mathfrak{a}$ such that $\varphi(x, y) = f([x, y])$. Then, define a map $\beta : \mathfrak{g} \to \mathfrak{h}$ by

$$\beta(x) = (x, f(x))$$

Clearly $p \circ \beta = 1_{\mathfrak{g}}$. Now, we define the following Lie bracket on $\mathfrak{h}$.

$$[(x_1, a_1)$$

For these reasons, we are interested in particular representations ρ of W such that the spectrum of $\rho(f_0)$ is non-negative (or at least bounded below). To simplify notation, we shall henceforth denote $\rho(f_n) = L_n$ for an arbitrary representation ρ.

Definition 3.1. A representation for which L_0 has a spectrum that is bounded below, ie. has a state of

Proof.
Let x_1, x_2 C. Then,

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} u, u & u, v \\ v, u & v, v \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} x_1 \ u, u + x_2 \ u, v \\ x_1 \ v, u + x_2 \ v, v \end{pmatrix}$$

$$= |x_1|^2 \ u, u + x_1 x_2 \ u, v + x_1 x$$

Proof.
Let n, m, k Z. As φ is a 2

We can characterize a highest weight representation of the Virasoro algebra by its central charge c and highest weight h. Hence, it is often convenient to write $V(c, h)$ for the Virasoro algebra. Furthermore, as h is

And as φ_0 annihilates each state, it evidently commutes with all φ_n and k. Hence, we have that $[\varphi_n, \varphi_m] = kn\delta_{n+m,0}$ and $[\varphi_n, k] = 0$

$$
\begin{aligned}
L_0|0\rangle &= \frac{1}{2k}\sum_{j \in \mathbb{Z}} \varphi_{-j}\varphi_j|0\rangle \\
&= \frac{1}{2k}\left(\sum_{j=0}^{\infty} \varphi_{-j}\varphi_j|0\rangle + \sum_{j=-\infty}^{-1} \varphi_{-j}\varphi_j|0\rangle\right)
\end{aligned}
$$

Then, replace i by $i + m$ in the second summation to obtain

$$[L_n, L_m] = \frac{n - m}{2k} \sum_{i \in \mathbb{Z}} \varphi_{-i}\varphi_{i+n+m}.$$

However, as $n + m = 0$ we have that $[\varphi_{-i}, \varphi_{i+n+m}] = 0$

7. Generalized Sugawara's Construction of the Virasoro Algebra

It is possible to derive the above construction of V

Using these definitions of the annihilation and creation operators, we find that the Hamiltonian H_k for the k^{th} degree of freedom is thus given by

$$H_k =$$