Differential Topology: A Survey

Cole Durham

1 Preliminaries

Differential topology serves in many ways as a bridge between many of the broad, global results of general topology and the local, analytical results of differential geometry. As we see in the foregoing work, fundamental quantities known to topology such as the Euler characteristic, singular homology, and the like, may be expressed in terms of fairly simple analysis. Work in this area was pioneered by notable mathematicians such as Poincare, Hopf, and Lefschetz, but much of our discussion will focus on contributions made by Marston Morse. Originally, Morse studied the general theory of calculus of variations, an area of mathematics quite commonly viewed only in relation to its physical applications in physics. However, although this topic seems removed from pure mathematics to the uninitiated, Morse's work ultimately showed that critical point theory, and hence differential topology, are at the heart of it all.

2 Background

To begin, we recall the extrinsic definition of a smooth manifold as a subset of Euclidean space as well as some of the rudimentary associated results. The most fundamental object in our study is the manifold, which we will define shortly. However, to give a precise notion of smoothness we first state

Definition 1 For open sets $U \subseteq \mathbb{R}^k$, $V \subseteq \mathbb{R}^l$, a map $f : U \to V$ is smooth if every partial derivative $\frac{\partial^m f}{\partial x_{i_1} \cdots \partial x_{i_m}}$ exist and are continuous. More generally, if $X \subseteq \mathbb{R}^k$ and $Y \subseteq \mathbb{R}^l$ are arbitrary subsets, then $f : X \to Y$ is called smooth if for $x \in X$, $\exists U \subseteq \mathbb{R}^k$ open with $x \in U$ and a smooth map $F : U \to \mathbb{R}^l$ such that $F|_{U \cap X} = f$.

Proof. Because f is a diffeomorphism, f^{-1} exists and is smooth, so $f^{-1} \circ f = Id_U$, therefore by (1) and (2) we know $d(f^{-1} \circ f) = d(f^{-1}) \circ df = Id_{\mathbb{R}^k}$, and similarly $df \circ d(f^{-1}) = Id_{\mathbb{R}^l}$. This implies df has a two-sided inverse, i.e. it is nonsingular, and then we must have

Theorem 1 (Sard) Let $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be smooth, U an open set. Denote by C the set of critical points of f, i.e. $C = \{x \in \mathbb{R}^m \mid rank(df_x) < n\}$. Then $f(C) \subseteq \mathbb{R}^n$ has Lebesque measure zero. [Interesting cases only when $m \geq n$, as $m < n \implies C = U$.]

To move to the more general setting of manifolds, we need only recall that a smooth manifold M is coverable by a countable collection of open sets, each of which is diffeomorphic to an open set $U \subseteq \mathbb{R}^m$. We immediately

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Claim: The projection $\pi_m : \mathbb{R}^m \to \mathbb{R}$ has 0 as a regular value.

To see this, note that for $x \in \pi_m^{-1}(0)$, $T_x(g^{-1}(y)$ is the null space of $df_x = df_x$. Our hypotheses stipulate that f has x as a regular point, as well as $f|_{\partial \mathbb{H}^m}$, so the null space is not completely contained in $\mathbb{R}^{m-1} \times \{0\}$. $\Rightarrow g^{-1}(y) \cap \mathbb{H}^m = f^{-1}(y) \cap U$. □

3 Intersection Numbers and Degrees

Throughout this section, the following setup will be used: (i) M is compact; (ii) $\partial M = \emptyset$; (iii) N is connected; (iv) $\dim M = \dim N$. These hypotheses are fairly restrictive, but they allow us to get to several results at the heart of differential topology.

3.1 Modulo 2 Version

Specific Example: To elucidate the functionality of the above definition, consider the standard basis in its usual order on $\mathbb{R}^n$: $\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}; \ldots; e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$. This gives the standard orientation; suppose instead we had the basis elements listed in the opposite order to give a new orientation $\{e'_1 = e_n; e'_2 = e_{n-1}; \ldots; e'_n = e_1\}$. Then notice $e'_{n+1-i} = \begin{bmatrix} 0 & & 1 \\ \vdots & \iddots & \vdots \\ 1 & & 0 \end{bmatrix} e_i$. If we denote this matrix by A, $\det A = (-1)^n$.

For familiarity, consider the familiar $\mathbb{R}^3$. Then we see that by labeling $x = e_1$, $y = e_2$, and $z = e_3$, we recover the notion of a "right-handed" or "left-handed" coordinate system, each of which corresponds to a distinct orientation of the space.

Definition 9 An <u>oriented manifold</u> M is one with a consistent choice of orientation on each $T_x M$

Proof. Orient $M \times I$ as a product; then $\partial(M \times I) = M \times \{0\} \cup M \times \{1\}$ where $M \times \{0\}$ will have the wrong orientation and $M \times \{1\}$ will have the correct orientation. Then by the previous Lemma, $deg(F|_{\partial(M \times I)}; y) = deg(g; y) - deg(f; y) = 0$, from which the result follows. □

With a Lemma in either hand, the Theorems of this section are proven in analogous fashion to their modulo 2 versions.

Application: Consider the family of maps $f_k : \mathbb{C} \to \mathbb{C}$ given by $f_k(z) = z^k$ for $k \in \mathbb{Z}$. In particular, we can easily see $f_k|_{S^1} : S^1 \to S^1$ maps the 1-sphere to itself as a k-fold covering. This map behaves nicely (even if $k < 0$, $(0,0) \notin S^1$), and we can compute that $deg(f_k) = k$. Unlike in previous sections when we could only have said $def(f_k) = k \mod 2$, which would only differentiate between even and odd values of k, integer degree theory separates all of these maps into distinct homotopy classes.

Moving forward, we wish to prove a general version of what is colloquially known as the Hairy Ball Theorem. To that end we must first make a definition.

Definition 11 A tangent vector field on $M \subset \mathbb{R}^k$ is a smooth assignment of vector $v : M \to \mathbb{R}^k$ such that $v(x) \in T_x M$ for all $x \in M$.

As previously mentioned, we are interested in the case $M = S^n$. The definition is satisfied there $\iff$ $(*)$ $v(x) \cdot x = 0$ for all $x \in S^n$. Moreover, if we *assume* $v($

is the indersection number of f with any point y; however, as before, we know this is independent of y.

This has some relatively simple intuition behind it: $graph(Id_X) =$, hence $L(Id_X) = I(;graph(Id_X)) =$ $I(;) =: (X$

except $N = (0,0,1)$ and $S = (0,0,-1)$ toward S. Then from a simple sketch we can see $L_N(f) = L_S(f) = +1$ (the fixed point at the north pole being a source, the south pole being a sink by construction). Hence $L(f) = \sum_{f(x)=x} L_x(f) = 2$. Clearly though a homotopy could be constructed between f and the identity, so from a previous theorem $L(f) = L(Id_{S^2}) = \chi(S^2) = 2$.

Corollary 13.1 The Euler characteristic of the 2-sphere is 2.

Corollary 13.2 Every map $f : S^2 \to S^2$ such that $f \simeq Id_{S^2}$ must possess a fixed point. In particular then the map $a(x) = -x$ cannot be homotopic to the identity.

Proof. Suppose such a map f did not possess a fixed point; then by the first theorem in this section, $L(f$

Recall the definition of the Gauss map, $g : X \to S^k$, $x \mapsto \hat{n}_x$ (changing to G& P notation here for unit normal vector at x). The Jacobian of this map is the <u>curvature</u> of X at x, $J(g) = \kappa(x)$. This data associated with X is strictly geometric (unlike the many topological structures we have seen), so it is not preserves by typical topological transformations. We can however say something beautiful about the total curvature, i.e. $\int_X \kappa$.

Gauss-Bonnet Theorem: If X is a compact, even-dimensional manifold embedded in $\mathbb{R}^{k+1}$ (a hypersurface), then

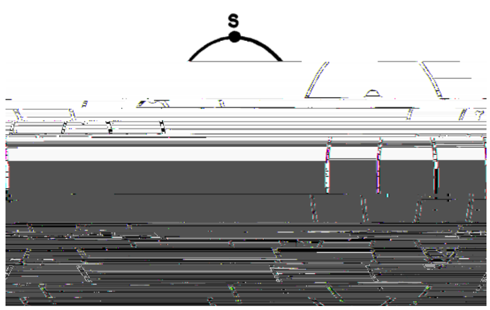
$$\int_X \kappa = \frac{1}{2} Vol(S^k) \, \chi(X).$$

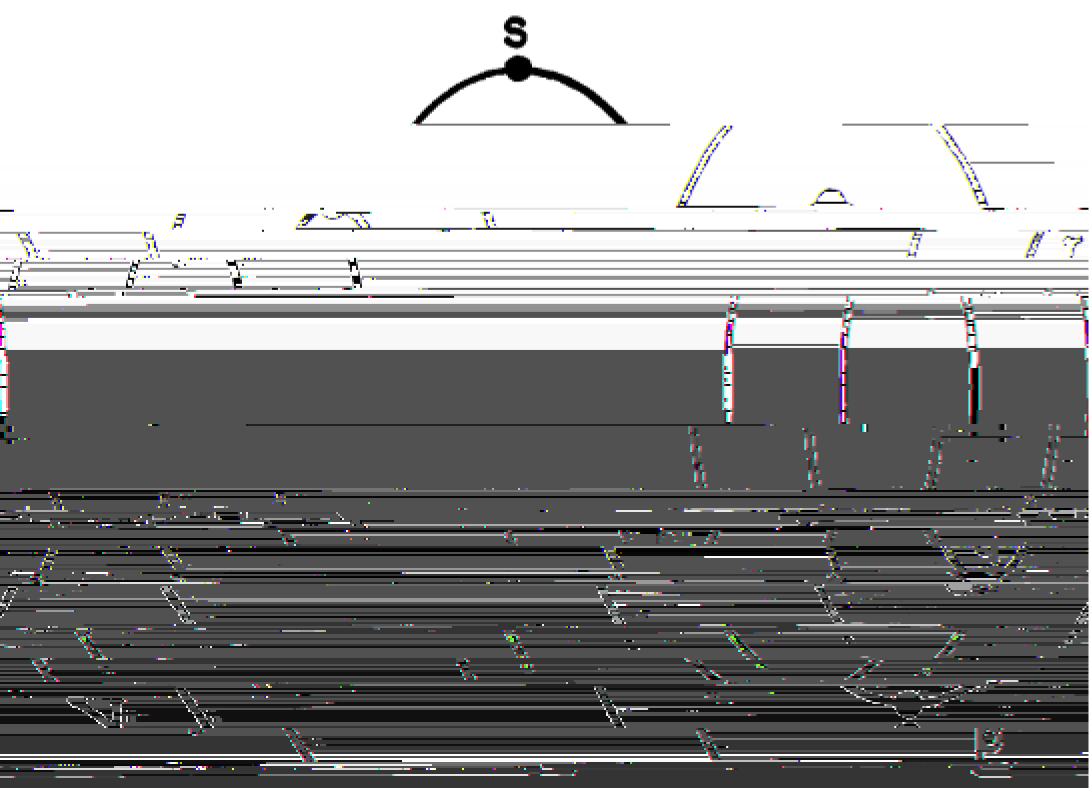
It is important to make note of the fact that the 'even-dimensional' hypothesis is of utmost importance; we saw in the previous section that if X were odd dimensional, it is automatic that $\chi(X) = 0$, but the integral of the curvature may not be.

7 Morse Theory

The previous sections give us a sense that topology may be approached through a variety of methods, a rather pleasant surprise. If possible we would like to know to what extend the topology of a manifold can be described by studying maps $f : M \to \mathbb{R}$. The amount of information which this theory, Morse theory, has to offer is tremendous. We introduce some of the main concepts and goals of through the following motivating example.

Example: Let $M = T^2$ as depicted below, tangent to the plane V.





This process of 'attaching' cells is terribly vague; let's make it more precise.
Let X

By inspection of the Hessian, we see there is a subspace $V \quad T_pM$, $\dim V = \quad$, on which $H_p(f)$ is negative de nite (span of rst columns of H). Similarly, there is a subspace W, $\dim W = n \quad$, on which $H_p(f)$ is positive de nite (span of last $n \quad$ columns of H). From dimensionality, if there were a subspace V^0 with $\dim V^0 > \dim V$ on which H was negative de nite, V^0 \

For a fixed value $q \in M$, consider the map given by $t \mapsto f(\varphi_t(q))$. Provided $\varphi_t(q) \in f^{-1}([a,b])$, the Fact from before says $\frac{d(f \circ \varphi_t(q))}{dt} = \left\langle \frac{d\varphi_t(q)}{dt}, grad\, f \right\rangle = \langle X, grad\, f \rangle = 1$. Thus we see this map is linear, with derivative 1 for $f(\varphi_t(q))$ between a and b.

Next, from this family $\{\varphi_t\}$, consider the diffeomorphism φ_{a-b}; Clearly this carries M^a diffeomorphically to M^b.

Lastly, define a 1-parameter family $r_t : M^b \to M^a$,

$$r_t(q) = \begin{cases} q; & f(q) \leq a \\ \varphi_{t(a-f(q))}(q) & \end{cases}$$

Inductively we have thus shown M^a for any a as speci ed above has the homotopy type of a CW-complex. If M

$$b \ (M^{a_k}; M^{a_0}) = b \ (M) \qquad \sum_{i=1}^{k} b \ (M^{a_i}; M^{a_i \ 1}) = c$$

where c is the number of critical points of index , since by above we know

$$H_i(M^{a_i}; M^{a_i \ 1}) = \begin{cases} R; & = \ _i \\ 0; & \text{else} \end{cases} \quad =) \ b \ (M^{a_i}; M^{a_i \ 1}) = \begin{cases} c \ ; & i = \\ 0; & \text{else} \end{cases} \quad :$$

In the case $f =$, which is additive, we see similarly

$$(M) = \sum_{i=1}^{k} (M^{a_i}; M^{a_i \ 1}) = \sum (\ 1)^i c_i:$$

We combine the previous results into one concise statement

Theorem 18 (Weak Morse Inequalities) If M is a compact manifold and the number of critical points of index on M is c , then (1) $b \ (M) \quad c$ and (2) $(M) = \sum (\ 1) \ b \ (M) = \sum (\ 1) \ c \ :$

Whenever possible, we want to sharpen inequalities. That is possible here based on the next Lemma.

Lemma 16 De ne the function F as

$$F \ (X; Y) = b \ (X; Y) \quad b \quad _1(X; Y) \qquad b_0(X; Y):$$

Then F is subadditive.

Proof. Consider the exact sequence of vector spaces given by

$$! \ V_0 \ ! \ V_1 \ ! \ V_2 \ ! \qquad ! \ V_n \ ! \ 0$$

where we denote each homomorphism mapping $! \ V_i$ by h_i

8 Morse Homology

Lemma 17 In local coordinates, $\nabla f = g^{ij}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}$.

Lemma 18 In local coordinates about a critical point, such that $\left\{\frac{\partial}{\partial x_i}\right\}$ is an orthonormal basis for the tangent space, the differential of the gradient is the Hessian, i.e. $\frac{\partial}{\partial x}\nabla f(p) = H_p(f)$.

Lemma 19 In local coordinates with the same orthonormal basis as before, the matrix for the differential of φ_t is given by

$$\frac{\partial}{\partial x}\varphi_t(p) = \exp(-H_p(f)t).$$

Proposition 14 If $f : M \to \mathbb{R}$ is Morse, where M is (among the hypotheses outlined at the beginning of the section) closed, then $M = \sqcup_{p \in Crit(f)} W^u(p)$, and/or similarly $M = \sqcup_{q \in Crit(f)} W^s(q)$.

A nice example: Consider the sphere $S^n \subset \mathbb{R}^{n+1}$ and define a map $f : S^n \to \mathbb{R}$ by $f(x_1, \dots, x_{n+1}) = x_{n+1}$ (if we assume the $n+1$ coordinate is the "vertical" one through the north and south poles N and S, this is the height function). It is clear f has 2 critical points, N and S, of index n and 0 respectively (since local coords about N would look like $-\sum x_i^2$, while about S they would look like $\sum x_i^2$). In turn, we find

$$W^s(N) = \{N\};\ W^u(N) = S^n - \{S\};\ W^s(S) = S^n - \{N\};\ W^u(S) = \{S\}.$$

8.2 Morse-Smale Functions

Definition 29 A Morse function $f : M \to \mathbb{R}$ satisfies Morse-Smale transversality if $W^u(q) \pitchfork W^s(p)$ for all critical points p, q of f. Such a function is simply called Morse-Smale.

From basic differential topology notions, embedded submanifolds $W(q; p) := W^u(q) \cap W^s(p)$ are obtained, and are of dimension $\lambda_q - \lambda_p$. As a consequence we find a Corollary.

Corollary 19.1 If $f : M \to \mathbb{R}$ is a Morse-Smale function, then the index of the critical points decreases strictly along gradient flow lines. In other words, if p, q are critical points of f with $W(q; p) \neq \emptyset$, then $\lambda_q > \lambda_p$.

Proof. Given $W(q; p)$ is nonempty, it contains at least one flow line from $q \to p$. The dimension of this flow line must be 1, hence $W(q; p) \geq 1$. □

Example Revisited: We return to the simple example of the height function of the torus positioned vertically on a plane. Immediately we notice a problem: the preceding Corollary does not appear to hold here, as the flow lines between the top of the "hole" and bottom of the "hole" connect critical points which are both of index 1, hence the index of the critical points is not strictly decreasing. This is because the standard height function is not Morse-Smale as the flow lines very clearly do not intersect transversely. We can however *tilt*

Corollary 20.1 If p, q are critical points with $\lambda_q - \lambda_p = 1$, then $\overline{W(q,p)} = W(q,p) \cup \{p, q\}$ and $W(q,p)$ has finitely many components (number of gradient flows from q to p is $< \infty$).

Proof. $W(q,p) \cup \{p,q\}$ since Corollary 19.1 indicates there are no intermediate critical points between q and p. As a result, $W(q,p) \cup \{p,q\}$ is a closed subset of a compact space, hence compact. We know the gradient flow lines between q and p form and open cover of $W(q,p)$ which can be extended to an open cover of $W(q,p) \cup \{p,q\}$. From the definition of compactness we thus see the number of gradient flow lines is finite. $\square$

Definition 32 For p, q critical points of $f : M \to \mathbb{R}$, p is an <u>immediate successor</u> of q if $q \succ p$ and $\nexists r \neq p$ such that $q \succ r$ and $r \succ p$.

Let $f : M \to \mathbb{R}$ be Morse-Smale and assume p is an immediate successor of q. Let $t \in \mathbb{R}$, be a regular value of f between $f(p)$ and $f(q)$. From the Regular Value Theorem the set $f^{-1}(t)$ for all such t is a submanifold of M, $\dim(f^{-1}(t)) = m - 1$, and $f^{-1}(t)$ is transverse to both $W^u(q)$ and $W^s(p)$. This leads to the following definition of two other submanifolds of M.

Definition 33 The <u>unstable sphere</u> of q is $S^u(q) = W^u(q) \cap f^{-1}(t)$, while the stable sphere of p is $S^s(p) = W^s(p) \cap f^{-1}(t)$, which are embedded submanifolds of dimensions $\lambda_q - 1$ and $m - \lambda_p - 1$.

Immediately following from this definition is the fact that $W^u(q) \pitchfork W^s(q) \implies S^u(q) \pitchfork S^s(p) \implies S^u(q) \cap S^s(p) =: N(q,p)$ is an embedded submanifold of dimension $_1$

With this number in mind we are finally able to construct the desired chain complex. The same hypotheses associated with M still apply.

Definition 35 Let $f : M \to \mathbb{R}$ be Morse-Smale and assume orientations for unstable manifolds associated with f have been chosen. Let $C_k(f)$ be the free abelian group of index k critical points of f and define $C_*(f) = \bigoplus_{k=0}^{m} C_k(f)$ where $m = \dim M$. The <u>Morse-Smale-Witten boundary operator</u> (abbreviated MSW) is a homomorphism $\partial_k : C_k(f) \to C_{k-1}(f)$ given by

$$\partial_k(q) = \sum_{p \in Cr_{k-1}(f)} n(q;p)p.$$

The pair $(C_*(f); \partial_*)$ is the <u>MSW chain complex of f</u>.

8.4 Morse Homology Theorem

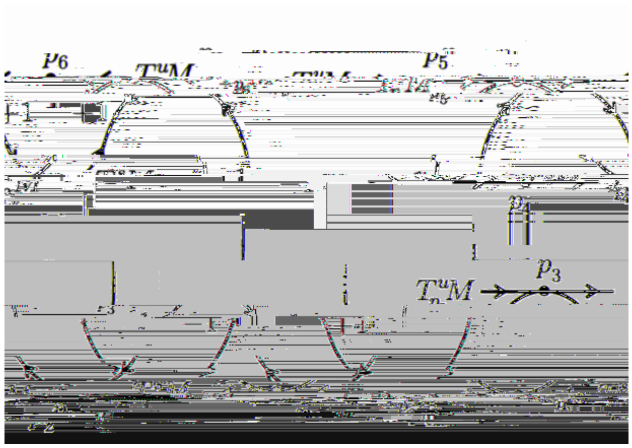
Theorem 21 The homology of the MSW chain complex $(C_*(f); \partial_*)$ is isomorphic to the singular homology $H_*(M; \mathbb{Z})$.

Remark 1 It is fairly intuitive to see the relation between the numbers $n(q;p)$ and the coefficient ring $\mathbb{Z}$ of $H_*(M; \mathbb{Z})$. However, for greater generality, the MSW chain complex can be constructed with coefficients in any commutative ring R via the tensor product $C_k(f) \otimes R$ and an application of the Universal Coefficient Theorem such that $(C_*(f); \partial_*) = H_*(M; R)$.

To motivate the Theorem, recall in Theorems 16 and 17 that we used a Morse function $f : M \to \mathbb{R}$ to prove any manifold M

From this we can compute directly $H_k(C_*(f);\partial_*) = ker(\partial_k)/im(\partial_{k+1})$ to find $H_k(C_*(f);\partial_*) = \begin{cases} \mathbb{Z}; & \text{if } k = 0,1 \\ 0; & \text{else} \end{cases}.$

Example 1a: Deformed S^1. Since homology should remain invariant under such transformations this is a valuable test. Again our function $f : S^1 \to \mathbb{R}$ is the height function but now we have many critical points p_i, $i = 1,\dots,6$. The image below provides all the orientation assignments that have been chosen



Following the same methods as when the circle was not deformed, we find the MSW chain complex looks like

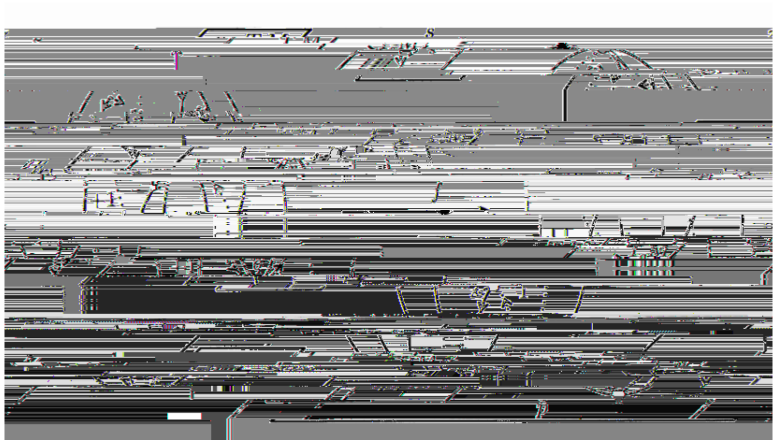
$$\begin{array}{ccccc} C_1(f) & \xrightarrow{\partial_1} & C_0(f) & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \\ \langle p_3,p_5,p_6\rangle & \xrightarrow{\partial_1} & \langle p_1,p_2,p_4\rangle & \longrightarrow & 0 \end{array}.$$

Now though, the boundary operator is a bit more difficult to describe, as we must compute $n(p_i,p_j)$ for pairs i,j ranging over $1,\dots,6$. These coefficients are arranged in the following matrix (a_{ij}) where $a_{ij} = n(p_i,p_j)$:

$$(a_{ij}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

We then can calculate $H_1(C_*(f);\partial_*) = ker(\partial_1)/0 = ker(\partial_1)$. Using the definition of the boundary operator, we know $\sigma = ap_3 + bp_5 + cp_6 \in ker(\partial_1)$ provided $\partial_1(\sigma) = 0$, i.e. $a\partial_1(p_3) + b\partial_1(p_5) + c\partial_1(p_6) = a(p_2 - p_1) + b(p_4 - p_1) + c(p_4 - p_2) = 0$. We see then if $a = -b = c$ that the equation is satisfied, hence $H_1(C_*(f);\partial_*) = ker(\partial_1) = \langle p_3 - p_5 + p_6\rangle \cong \mathbb{Z}$. Similarly, $H_0(C_*(f);\partial_*) = ker(\partial_0)/im(\partial_1) = \langle p_1,p_2,p_4\rangle/\langle p_2 - p_1, p_4 - p_1, p_4 - p_1\rangle = \{(p_1,p_2,p_4) \mid p_1 = p_2 = p_4 \in \mathbb{Z}\} \cong \mathbb{Z}$. It is clear the homology groups for $k \neq 0,1$ are 0, hence we arrive again at the conclusion $H_k(C_*(f);\partial_*) = \begin{cases} \mathbb{Z}; & \text{if } k = 0,1 \\ 0; & \text{else} \end{cases}$ as desired.

Example 2: Tilted T^2



As seen prior, there are four critical points p, q, r, s of index $0, 1, 1, 2$ respectively. The resulting chain complex looks like

$$\begin{array}{cccc} C_2(f) \xrightarrow{\partial_1} & C_1(f) & C_0(f) & 0 \\ \downarrow \cong & & & \\ \langle s \rangle & \langle q, r \rangle & \langle p \rangle & 0 \end{array}$$

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