Dynamics of Rotation-like Logistic Maps

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Abstract

In this paper, we study the dynamics of rotation-like logistic maps. In particular, we focus on the case of rotation-like logistic maps with Fibonacci quotient, including related parts on Hubbard tree, kneading invariant and kneading map. We shall present theorems and conjectures on their associated kneading maps.

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2 Introduction

At the beginning of the twenty-first century, Blé and Douady [2] introduced a family of logistic maps inspired from ideas originated from holomophic dynamical systems. Despite the connection between the dynamics of logistic maps and holomorphic dynamics, there has been little study carried on from the perspective of both areas. Our goal is to study the kneading invariant and the kneading map of the logistic maps in Blé-Douady's family. In this paper, we shall only study one particular subset of these logistic maps: rotation-like logistic maps with Fibonacci quotient.

3 Preliminaries

3.1 Basic Concepts in Complex Dynamics

A **discrete dynamical system** (X, f) is a space/set X along with a map $f : X \to X$, where X is our "system", and f is "the law of dynamics".

Definition If $f(z) = z$, then z is called a **fixed point** of f. More generally, if $f^n(z) = z$ and $f^m(z) \neq z \, \forall 1 \leq m \leq n-1$, then $z = z_0 \to f(z) = z_1 \to f^2(z) = z_2 \to \cdots \to f^{n-1}(z) = z_{n-1}$ is called a **periodic orbit** of period n.

Definition Let z be a fixed point of f. If $\exists U$, a neighborhood of z such that $f(U) \subset U$ and $\forall z' \in U$, we have $\lim_{n \to \infty} f^n(z') = z$, then z is called an **attracting fixed point**. If instead, z is in an periodic orbit of period n, then $z = z_0 \to f(z) = z_1 \to f^2(z) = z_2 \to \cdots \to f^{n-1}(z) = z_{n-1}$ is called an **attracting periodic orbit** if z is an attracting fixed point of f^n.

ducing another special set:

Theorem (Fatou) Let p be a non-linear polynomial of order n. Then every immediate basin of attraction for a periodic orbit contains at least one critical point of p.

Corollary There can be at most $n - 1$ different attracting periodic orbits.

As a corollary in our quadratic polynomial case, the immediate basin of attraction of an attracting periodic orbit, which has to be bounded (if $|z|$ is large enough, $p_c^n(z)$ will blows to infinity no matter what c we choose), must contain the orbit $\{p_c^n(0)\}$. Inspired by this observation, we now define the following more general set, called the **Mandelbrot set**:

$$M = \{c : \exists A_c, |p_c^n(0)| < A_c, \forall n \in \mathbb{N}\} \tag{2}$$

It is conjectured that M is the closure of M_0. We shall not discuss further on that.

It is proved by Douady and Hubbard that the Mandelbrot set is connected.

Following from our previous discussions, there might also be of some interest to consider the collection of all points that are bounded under infinite iteration. It is called the **filled Julia set**, denoted by $K = K(f)$. The formal definition is

$$K(f) = \{z : \exists C_z > 0, f^n(z) < C_z, \forall n \in \mathbb{N}\} \tag{3}$$

Its boundary ∂K is called the **Julia set**, denoted by $J = J(f)$. We see that Julia set is indeed the boundary between being "blowing to infinity" and "forever bounded" after infinite iterations.

For a quadratic family $p_c(z) = z^2 + c$, we denote K_c to be $K(p_c)$. Actually, there is another version of the definition Mandelbrot sets, which is the primary version used in Ble's paper. Proving equivalence to our previous definition is very nontrivial and involves the use of Green function, which is somewhat a characterization of \the speed of blowing to in nity". We shall not discuss into details here.

De nition' The Mandelbrot set M is the collection of all c's for which K_c is connected.

There are some very interesting properties of Julia sets. We will introduce them without detailed proofs:

Theorem If f is a rational function, then $J(f)$ 6= ;.

Theorem J is forward invariant. More speci cally, let z 2 $\hat{C}$, then $f(z)$ 2 J if and only if z 2 J.

Theorem $J(f) = J(f^n)$8n 2 N.

There are two equivalent de nitions of **Fatou set**. One of them is more \intuitive", and the other is de ned using the normal family of functions.

De nition The Fatou set is the complement of the Julia set. In other words, it is the union of all points that are bounded under in nite iterations of f, as well as those in the basin of attraction for 1 .

The following theorem is a great illustration of what Julia and Fatou sets really mean:

Theorem The repelling orbit points of f are completely contained in $J(f)$; on the other hand, the attracting orbit points of f are completely contained in $\hat{C}$ n $J(f)$.

Let us recall how we constructed the Mandelbrot set M. The set in the rst iteration, consisting of all the values of c resulting in an attracting xed point, denoted by M_1, **main hyperbolic component**, and @M_1 is called the **main cardioid**.

A point in M with an attracting cycle is called **hyperbolic**.

Theorem M is locally connected when restricted to the hyperbolic components.

In particular, M is locally connected when restricted to the main cardioid.

Definition The orbit of a critical point with respect to a polynomial f is called a **critical orbit**. The critical orbit is **preperiodic** if the orbit is finite. If the critical orbits are periodic or preperiodic (or both), then the polynomial f is called **PCF (postcritically finite)**. In the case that all critical orbits are periodic, f is called a **center**; if they are all preperiodic, f is called a **Misiurewicz polynomial**. Denote the set of all critical points of f by $C(f)$. Then the **postcritical set** is the union of all subsequent images of all critical points under f.

Definition Let X, Y be topological spaces and let $(X; f)$ and $(Y; g)$ be their corresponding dynamical systems. If h is a homeomorphism from Y to X such that $h^{-1} \circ f \circ h = g$, then $(X; f)$ and $(Y; g)$ are called **topologically conjugate**. If instead h is continuous and surjective (but not necessarily a homeomorphism), then $(X; f)$ and $(Y; g)$ are called **semi-conjugate**.

In some sense, we see that conjugation means "the same after a change of coordinates". Theorems by Koenig and Bottcher show that a polynomial map behaves "locally like a linear or a monomial map" near its attracting fixed point in the sense that the map is conjugate to a polynomial in some open neighborhood of its fixed point under a conformal map. Therefore, the dynamics (i.e. the properties of the corresponding maps under long iterations) of two conjugate dynamical systems are very similar.

It can be shown that when restricted to $J(p_c)$, p_c is semi-conjugate to the angle doubling map $: \mathbb{T} \to \mathbb{T} : \ \mapsto 2$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a representation of the circle with unit circumference. In addition, our choice of can ensure that the orbit of under the angle doubling map does not intersect the interval $[\frac{1}{4} + \frac{}{4}; \frac{1}{2} + \frac{}{4}]$.

3.2 Hubbard Tree

Now we construct the Hubbard tree (this follows the definition in Ble's paper [2]). A graph $= (V; E)$ consists of a set of vertices V and a set of edges E, each connecting two vertices. If in addition, is connected (intuitively, you can start from any vertex and reach any other vertex by going through the edges in E) without a loop, then it is called a **tree**. For convenience, if n edges meet at the same vertex, then the angle between any two edges is an integer multiple of $\frac{1}{n}$.

Definition A **Hubbard tree** is a tree that further satisfies the following properties:

(1) There exists a skew-symmetric function $: (l; l') \to \mathbb{T}$, assigning each pair of edges meeting at the same vertex an angle, that satisfies ($(l; l') = 0$ if and only if $l = l'$) and ($(l; l') + (l'; l'') = (l; l'')$).

(2) There exists a map $: V \to \mathbb{N}$, called a **local degree function**, that assigns a *degree* to each vertex, with the property $1 + \sum (\ (v) \ \ 1) > 1$. (Here, $\mathbb{N}$ does not include 0.) If $(v) > 1$, then v is **critical**. Note that if there is no critical vertex, then $1 + \sum (\ (v) \ \ 1) = 1 + 0 = 1$, so there must be at least one critical point.

(3) There exists a homeomorphism $: H \to H$ mapping vertices to vertices and edges to edges, with the property that $(\ (l);\ (l')) = \ (v)\ (l; l')$ where v is the vertex on which the two edges meet. If $^n(v) = v$ for a positive integer n, then v is called a **periodic**. If v is a critical point in addition, then its orbit is called a **critical cycle**. If v is not in the mage of a critical cycle under $^n \forall n \in \mathbb{N}$, the v is a **Julia vertex**; otherwise, v is called a **Fatou**

vertex.

(4) There exists a metric $d : V \times V \to \mathbb{N} \cup \{0\}$, counting the number of edges in the shortest path between two vertices.

(5) H is **expanding**. This means that for all Julia vertices $v; v'$ that are connected by an edge l, $d(\ ^{n}(v);\ ^{n}(v')) > 1$ for some n_l.

Here we provide three concrete examples for the Hubbard tree. Let us consider a **rotation-like logistic map** shorter half-circle (unless $= 1/2$, which we shall not consider here). Just for mathematical rigor, take the closure of the two half-circles so they are both compact segments. Relabel x_1 and x_2 on the longer half-circle as x_1' and x_2'.

3. “Paste” x_2 and x_1' together (more rigorously, we can consider performing a quotient map by identifying these two points on the two half-circles) and label it as . (We do not have to use in our discussion below. This is just for construction.)

4. Now we have an interval. The interval goes from x_1 to x_2' in ascending order. We can map this interval to the real line homeomorphically and set $x_0 = 0$ for simplicity.

We can see the first three examples in the picture below:

Now that we have a Hubbard tree, we are looking for a map such that $f(x_0) = x_1$, $f(x_1) = x_2$, etc. It is not hard to imagine that there are many maps that satisfy this condition, even if we stipulate that the map must be holomorphic.

Theorem A Hubbard tree corresponds to a unique PCF (a polynomial map that have finitely many critical points along with their iterations under f) up to an affine conjugation.

This theorem is critical in that we can construct our Hubbard trees with more freedom in choosing a corresponding map.

We now introduce the notion of kneading invariant.

3.3 Kneading Invariant

Suppose we have a **unimodal map** $f : [0,1] \to [0,1]$, which means that f is a continuous map with one and only one maximum x_0 and that $f(0) = 0 = f(1)$. Let x_0^+ be the right limit of x_0.

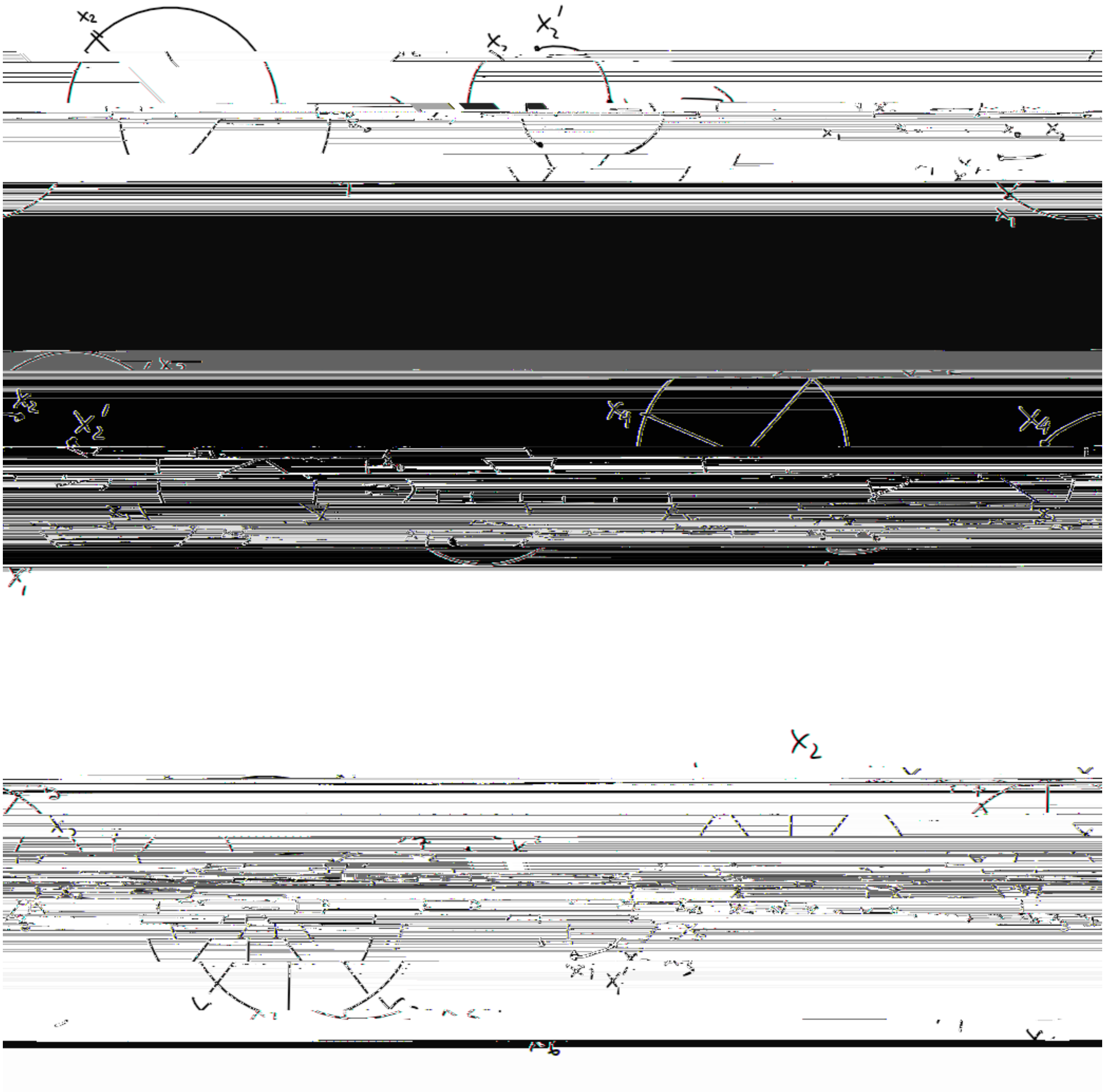

Figure 1: Examples on construction of the Hubbard Tree

Definition The **kneading invariant** of f is defined as a sequence $v(f) = \overline{v_1 v_2 v_3 \quad}$, where each v_i denote the i-th digit in the sequence, and is defined to be:

$$v_i = \begin{cases} 0; & \text{if } f^i(x_0^+) < x_0 \\ 1; & \text{if } f^i(x_0^+) > x_0 \end{cases}$$

Here, we use x_0^+ instead of x_0 in order to avoid the case when $f^{(i)}(x_0) = x_0$.

Also, we are only considering the case where we are in the "dynamical core". In other words, $f^{(2)}(x_0) < x$

into an infinite number of blocks in a unique way through an algorithm.

The algorithm for the decomposition is as follows [8]:

1. Start with either 0 or 1, depending on whether $v(f)$ starts with 0 or 1, and consider $0v(f)$ or $1v(f)$ accordingly.

3.4 Kneading Map

The kneading map, compared to kneading invariant, is defined in a much more implicit way.

Definition The **kneading map** $Q : \mathbb{N} \to \mathbb{N}$ is the map that satisfies:

$$R_{Q(n)} + 1 = r(n) \tag{4}$$

The image of the kneading map of a rotation-like logistic map with Fibonacci quotient is a repetitive sequence. In other words, there is some period p such that $Q(N) = Q(N \bmod p)$. This directly follows from the fact that the kneading invariant of f is periodic, since this means the block decomposition of) is 7.040504)

Definition. The **kneading map** $Q:\mathbb{N}\to\mathbb{N}$ is the map that satisfies:

$$S_{Q(n)} = S_n - S_{n-1} \tag{5}$$

4 Examples

We now compute two examples of the kneading invariant of the logistic maps with Fibonacci quotient. Before that, we need to have a map which takes x_0 to x_1, x_1 to x_2, $\ldots$. This map does not have to be unique; rather, as we have previously discussed, they only have to be equivalent up to an affine conjugation. In this case, just for simplicity, we use the quadratic map $f(x) = x^2 + c$. Now, a requirement is that when iterated enough times (in particular, multiples of the period of the number of points in the Hubbard tree), f takes x_0 back to itself. Here, we assume that the points in the Hubbard tree are mapped to the real line and let $x_0 = 0$ be the turning point of the unimodal map f. Actually, f is not exactly a unimodal map here, since it is not mapping from $[0,1]$ to $[0,1]$. However, it is equivalent to a unimodal map up to an affine conjugation, so they have the same dynamics and therefore can be treated equally. We are mapping the Hubbard tree to a real interval, and it does not matter much where the interval is on the real line regarding the dynamics on the Hubbard tree.

4.1 Case $\theta = \frac{3}{5}$

We can actually solve for c by noting that $f(x_0) = f(0) = c = x_1$, $f(x_1) = c^2 + c = x_2$, $f(x_2) = (c^2+c)^2+c = x_3$, $f(x_3) = ((c^2+c)^2+c)^2+c = x_4$, $f(x_4) = (((c^2+c)^2+c)^2+c)^2+c =$

$x_0 = 0$. Although the value of c does not matter much in our discussion later, we will just compute c as a reference. Note that since this is a polynomial, we will have multiple solutions of c, which may or may not be real, and not all of them will satisfy the requirement that the relative position of the images of x_0 must be the same as we desired. For example, the trivial solution $c = 0$ is not valid for our purposes, since if we take $c = 0$, we will have $x_0 = x_1 = x_2$ instead of $x_1 < x_0 < x_2$. An unproved conjecture is that only one c is valid. The only valid c here is c 1.6254.

The kneading invariant is 01000 . The repeating part is 01000. Let us practice the block decomposition for this map as an example:

1.Starting with the first digit 0 and comparing the sequence of $v(f)$ starting from the first digit with $0v(f)$ (recall that if the digit we are considering is 0, we compare the kneading invariant starting with this digit with $0v(f)$; otherwise, we compare it with $1v(f)$), since 1 is the second digit in $v(f)$, and 0 is the first digit of $v(f)$, and $1 \neq 0$, we stop, so the first block is (0).

2.Next, starting from the second digit 1 and comparing the portion of $v(f)$ starting from the second digit with $1v(f)$, since the third digit in $v(f)$ is 0, which is equal to the first digit in $v(f)$, we continue; comparing the fourth digit of $v(f)$, which is 0, with the second digit of $v(f)$, which is 1, we find $0 \neq 1$, so we stop on this block, and the the second block is (10).

3.Next, starting from the fourth digit of $v(f)$, which is 0, and compare the sequence of $v(f)$ starting from it with $0v(f)$: since the fifth digit of $v(f)$ is 0, and the first digit of $v(f)$ is 0, we continue; now, the sixth digit of $v(f)$ is 0, and the second digit of $v(f)$ is 1, $1 \neq 0$, so we stop with this block, so the third block is (00).

4.Since the kneading invariant forward is just repeating the first five digits, the block decomposition will be exactly the same as the three we did above, which means we will have blocks (0)(10)(00) repeating forever.

Now we calculate the sequence of kneading map of f:

1.First, $R(Q(1)) + 1 = r(1) = 1$, so $R(Q(1)) = 0$. Since R is a strictly increasing function (a block must have length at least 1), and $R(0) = 0$, we must have $Q(1) = 0$.

2.Next, $R(Q(2)) + 1 = r(2) = 2$, or $R(Q(2)) = 1$. By the same reason, since $R(1) = 1$, we must have $Q(2) = 1$.

3.Next, $R(Q(3)) + 1 = r(3) = 2$, or $R(Q(3)) = 1$. By the same reason, since $R(1) = 1$, we must have $Q(3) = 1$.

4.From this point on, for any j 4, $R(Q(j \bmod 3)) + 1 = r(j \bmod 3)$, so we must have $Q(j) = Q(j \bmod 3)$.

4.2 Case $= \frac{5}{8}$

Similarly as the procedures in the last example, we have $f(x_0) = f(0) = c = x_1$, $f(x_1) = c^2 + c = x_2$, $f(x_2) = (c^2 + c)^2 + c = x_3$, $f(x_3) = ((c^2 + c)^2 + c)^2 + c = x_4$, $f(x_4) = (((c^2+c)^2+c)^2+c)^2+c = x_5$, $f(x_5) = ((((c^2+c)^2+c)^2+c)^2+c)^2+c = x_6$, $f(x_6) = (((((c^2+c)^2+c)^2+c)^2+c)^2+c)^2+c = x_7$, $f(x_7) = ((((((c^2+c)^2+c)^2+c)^2+c)^2+c)^2+c)^2+c = x_0 = 0$. By checking the relative positions of the images of x_0 under f for different c, we see that the only valid c here is c 1.7111.

The kneading invariant is 01001000 . The repeating part is 01001000. Let us practice the block decomposition for this map:

1.Starting with the first digit 0 and comparing the sequence of $v(f)$ starting from the first digit with $0v(f)$ (recall that if the digit we are considering is 0, we compare the kneading invariant starting with this digit with $0v(f)$; otherwise, we compare it with $1v(f)$), since 1 is the second digit in $v(f)$, and 0 is the first digit of $v(f)$, and $1 \neq 0$, we stop, so the first block is (0).

2.Next, starting from the second digit 1 and comparing the portion of $v(f)$ starting from the second digit with $1v(f)$, since the third digit in $v(f)$ is 0, which is equal to the first digit in $v(f)$, we continue; comparing the fourth digit of $v(f)$, which is 0, with the second digit of $v(f)$, which is 1, we find $0 \neq 1$, so we stop on this block, and the the second block is (10).

3.Next, start from the fourth digit 0, and compare the sequence of $v(f)$ starting from it with $0v(f)$: since the fifth digit of $v(f)$ is 1, and the first digit of $v(f)$ is 0, we stop here, so the third block is (0).

4.Next, start from the fifth digit 1, and compare the sequence of $v(f)$ starting from it with $1v(f)$: since the sixth digit of $v(f)$ is 0, which is equal to the first digit 0, we continue; since the seventh digit is 0, which is not equal to the second digit 1, we stop here, so the fourth block is (10).

5.Next, start from the seventh digit 0, and compare the sequence of $v(f)$ starting from it with $0v(f)$: since the eighth digit 0 is equal to the first digit 0, we continue; since the ninth digit 0 is different from the second digit 1, we stop here, so the fifth block is (00).

6.Since the kneading invariant forward is just repeating the first eight digits, the block decomposition will be exactly the same as the three we did above, which means we will have blocks (0)(10)(0)(10)(00) repeating forever.

Now we calculate the sequence of kneading map of f:

1.First, $R(Q(1)) + 1 = r(1) = 1$, so $R(Q(1)) = 0$. Since R is a strictly increasing function (a block must have length at least 1), and $R(0) = 0$, we must have $Q(1) = 0$.

2.Next, $R(Q(2)) + 1 = r(2) = 2$, or $R(Q(2)) = 1$. By the same reason, since $R(1) = 1$, we must have $Q(2) = 1$.

3.Next, $R(Q(3)) + 1 = r(3) = 1$, or $R(Q(3)) = 0$. By the same reason, since $R(0) = 0$, we must have $Q(3) = 0$.

4.Next, $R(Q(4)) + 1 = r(4) = 2$, or $R(Q(4)) = 1$. By the same reason, since $R(1) = 1$, we must have $Q(4) = 1$.

5.Next, $R(Q(5)) + 1 = r(5) = 2$, or $R(Q(5)) = 1$. By the same reason, since $R(1) = 1$, we must have $Q(5) = 1$.

6.From this point on, for any j 6, $R(Q(j \bmod 6)) + 1 = r(j \bmod 5)$, so we must have $Q(j) = Q(j \bmod 5)$.

5 Main Conjectures

For a rotation-like logistic map $f_n = e^{2\ i\ n}$ with Fib)with Fib) -36e

(ii) $v_{F_k} = 0$ if k 0 mod 2;

(iii) $v_{F_{n+1}} = 0$;

References