THE TOPOLOGY OF MAGMAS

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Introduction

A *magma* is an algebraic structure (S, f) consisting of an underlying set S and a single binary operation $f : S^2 \to S$. Much is known about specific families of magmas (semigroups, monoids, groups, semilattices, quasigroups, etc.) as well as magmas in general as treated in universal algebra. We seek to relate the study of magmas to the study of corresponding geometric objects. In order to do this we first analyze unary operations by way of their graphs. We show how function composition can be encoded by matrix multiplication, then generalize this to binary function composition. We characterize the spectra of the graphs of unary operations, show that all such graphs are planar, and present some initial results on the corresponding constructions for magmas.

Contents

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There are two competing conventions here. Usually when we regard matrices as linear transformations we think of them as mapping column vectors on the right into row vectors. Graph theory indicates the opposite behavior, with function application occurring on the right.

1.3. **Graph Treks.** Recall that given a graph $G = (V, E)$, which need not be simple and may be directed, we have the following theorem.

Theorem. *Let A be the adjacency matrix for G with a given vertex ordering. Then $(A^k)_{ij}$ for k N is the number walks of length k from v_i*

in a given column j giving the total number of valid treks beginning at any vertex s_i and ending at s_j, which is also the number of solutions $x = s_i$ to $f^Q(x) = y$ for a fixed $y = s_j$. We can take the total succinctly by summing over all rows i, so the number of solutions x is □

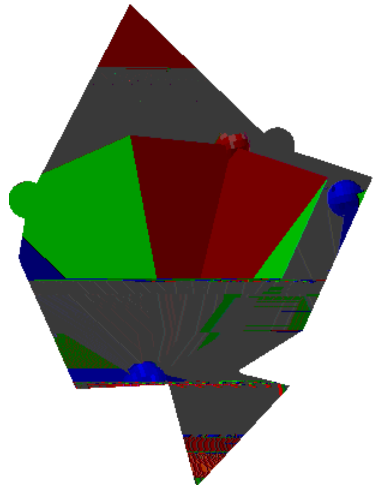
2.1. **Operation Hypergraphs.** We can view a binary operation as a set

$$\{(s_i, s_j, f(s_i, s_j)) \mid s_i, s_j \quad S\}.$$

This set can be seen as the edge set of a directed 3-uniform hypergraph[1].

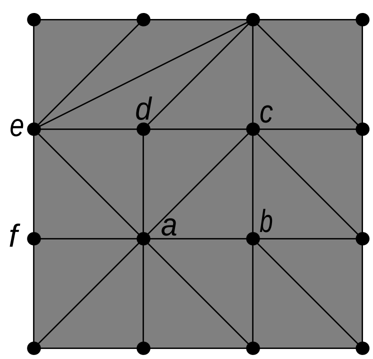
Definition (Operation hypergraph). Let $f : S^2 \to S$ be a binary operation. The

that $\bar{G}_f$ contains a subgraph H which is a subdivision of K_5. This subgraph contains five vertices, say s_1



Again let (S, f) be a magma. We demonstrate a technique for generating algebraic conditions which imply that the embedding dimension of (S, f) is at least 4. Recall that the Klein bottle cannot be embedded in $\mathbb{R}^3$ without self-intersection. We know the minimal triangulations of the Klein bottle[15] so we can orient such a triangulation to obtain a minimal algebraic rule which implies that a given magma has embedding dimension at least 4.

Consider the triangulation Kh12 from [15], which is pictured below. The horizontal edges are to be identified in parallel and the vertical edges in antiparallel.



	a	b	c	d	e	f	g	h	i
a	$\cdot$	c	d	e	f	g	h	i	b
b	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
c	$\cdot$	$\cdot$	$\cdot$	i	b	$\cdot$	$\cdot$	$\cdot$	
d	$\cdot$	$\cdot$	$\cdot$	$\cdot$	i	$\cdot$	$\cdot$	$\cdot$	
e	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	c	b	$\cdot$	$\cdot$
f	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	i	$\cdot$	c
g	$\cdot$	$\cdot$	$\cdot$	$\cdot$	h	$\cdot$	$\cdot$	$\cdot$	b
h	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	e
i	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

This "forbidden substructure" cannot appear in any magma with embedding dimension 3. We can extend our earlier example of magmas with embedding dimension 3 to produce a magma with embedding dimension 4. For each pair x, y for which $\cdot$ appears in the table above define $f(x, y) = x$. None of these new degenerate faces will change the embedding dimension of the magma, so the resulting operation has embedding dimension 4.

It is immediate that embedding dimension can only decrease when considering a submagma of a given magma. What relationship does embedding dimension have with taking homomorphic images and products of magmas? If it only goes down then we know that "magmas of embedding dimension at most k" is a variety and hence an equational class by Birkhoff's Theorem[2]. This would tell us that there is a set of identities which characterize such magmas (and hence their operation complexes). If not, we can show that it is impossible to produce such a characterization.

3.2. Spectrum Calculation.

There is a very direct relationship between the spectrum of an operation digraph and the dynamics of the original function.

Theorem. *Let $f : S \to S$ be a function on a set S of size n. Let $c_1, \dots, c_r$ be the lengths of the periodic cycles of S under f, with multiplicity. For each c_i the matrix A_f has all of the c_i^{th}*

point of S after n applications of f, we see that

$$A_f^{n!} = \begin{bmatrix} I & O \\ C & O \end{bmatrix}$$

where I is the $k \times k$ identity matrix and C is some other matrix.

Given such a lower triangular matrix we have that $\det A_f^{n!} = (\det I)(\det O)$ This implies that

$$\det(\lambda I - A_f^{n!}) = (\lambda - 1)^k \lambda^{n-k},$$

so the spectrum of A consists of k roots of unity and must be the spectrum of A_f, as well. □

In contrast with this complete description of the spectrum of an operation digraph, no such generic description of a spectrum for a uniform hypergraph is known to this author. A possible future project is to use the special case of operation hypergraphs as a stepping stone to the general case.

References

[1] Claude Berge. *Hypergraphs: Combinatorics of Finite Sets*. Elsevier Science Publishers B.V., 1989. isbn: 0444874895 (cit. on p. 8).

[2] Clifford Bergman. *Universal Algebra: Fundamentals and Selected Topics*. Chapman and Hall/CRC, 2011. isbn: 978-1-4398-5129-6 (cit. on p. 15).

[3] J. R. Doyle. “Preperiodic points for quadratic polynomials with small cycles over quadratic fields”. In: