KNEADING INVARIANTS OF FIBONACCI QUOTIENT ROTATIONS

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Abstract. This paper studies the sequence of rotations of the unit circle by angles of quotients of adjacent Fibonacci numbers. An equivalence relation is placed on the unit circle that converts it into an interval in the real line. On this interval, each Fibonacci quotient rotation forms a real, quadratic, one dimensional dynamical system. We give an inductive construction for the kneading invariants of this sequence of dynamical systems.

1. Introduction

We consider the quadratic family of polynomials, $P_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by $P_c(z) = z^2 + c$, for $c \in \mathbb{C}$. Given $c \in \mathbb{C}$, we define the *filled Julia set*, K_c as those $z \in \mathbb{C}$ such that the orbit of z by P_c, $\{P_c^n(z) : n \in \mathbb{N}\}$, is bounded and Julia set as the boundary of the filled Julia set, ∂K_c.

The Mandelbrot set, M, is the set

$$M = \{c \in \mathbb{C} : K_c \text{ is connected}\}.$$

It is known that M is compact and connected [3]. By the Riemann mapping theorem, there is an analytic homeomorphism $\Phi_M : \mathbb{C} \setminus K \to \mathbb{C} \setminus \overline{\mathbb{D}}$ (where $\mathbb{D}$ is the unit disk). Define the *external ray of argument*

The proof makes use of a Hubbard tree. A tree is an finite, connected, and acyclic graph embedded in $\mathbb{C}$. A Hubbard tree of a given map P_c is a tree, $T \subset K_c$, such that T contains the orbit of 0 and no subtree T' both contains the orbit of 0 and has that $P_c(T') \subset T$. Hubbard trees distill all the combinatorial information of the map they represent into a simple structure. We can generalize Hubbard trees with the following definition. An *abstract Hubbard Tree* is a tree with a continuous and onto map g such that: g is at most two to one, except for a single point, called the *critical point*, g is a local homeomorphism, and every endpoint of T lies on the forward orbit of the critical point [2]. An abstract Hubbard Tree is said to be expanding if for each edge with endpoints $v_1; v_2$ there is an $n \in \mathbb{N}$ such that the number of edges between g

The goal of dynamical systems is to describe how the orbits are distributed. If there is $n \in \mathbb{N}$ such that $f^n(x) = x$ then we say that x is n-periodic and $O_f(x)$ is a periodic orbit.

A continuous map f defined on an interval $I = [a, b]$ is called unimodal if there is $c \in (a, b)$ such that f is strictly monotone on $[a, c)$ and $(c, b]$ and c is a global maximum or minimum. The point c is called the turning point of f. For an unimodal map f with turning point c, write $c_i = f^i(c)$ [2].

2.2. Kneading Theory. Given a dynamical system (f, I) where f is an unimodal map and a point $x \in I$ we define the itinerary of f and x, $I(f, x)$, to be the sequence $e_1, e_2, \ldots$ where

$$e_i = \begin{cases} 0 & f^i(x) < c \\ 1 & f^i(x) > c \\ \star & f^i(x) = c \end{cases}$$

If the last case occurs, the itinerary is finite and stops before the star. In other words if x is n-periodic then $I(x, f)$ is a finite sequence of $n-1$ binary values. We call $I(c_1, f)$ the kneading sequence or the kneading invariant of f [2].

2.3. Diophantine Approximations.

By rewriting we get

$$= 1 + \frac{1}{}: \tag{3}$$

Thus for all n 2 N_0, $_n$ = and a_n = b c = 1. Hence p_1 = 1, p_2 = 2 and in general $p_n = p_{n\ 1} + p_{n\ 2}$. In particular $p_n = F_{n+1}$. Similarly $q_n = F_n$ for all n 2 N_0. Plugging this into 1 and 2 we get the following equations

$$F_{n+1} \quad F_{n+2} = \frac{(\ \ 1)^{n+1}}{F\ _{n+1} + F_n} \tag{4}$$

$$F_n \quad F_{n+1} = \frac{(\ \ 1)^{n}}{F\ _{n+1} + F_n}: \tag{5}$$

We will now prove a stronger version of Theorem 6 in Chapter 1, Section 2 of [6] for the speci c case of . Our proof follows the same technique as the proof in [6] but changes some inequalities. First, we make the following change to the de nition of a best approximation appearing in [6]. We say that a=b with a; b 2 N is a best approximation to if for all b^0 2 N with 1 $b^0 < b$,

$$kb\ \ k = jb \quad aj \quad \text{and} \quad \frac{kb^0\ \ k}{b^0} > \frac{kb\ \ k}{b} \tag{6}$$

Theorem 6 ([6]). The best approximations to are its principal convergents. In particular, for n 2 N, F_n is the smallest integer $q > F_{n\ 1}$ such that $kq\ \ k=q < kq_n\ \ k=q_n$.

Proof. First we will show that if a=b 2 Q is a best approximation with a; b 2 N then there is n 2 N such that a = F_{n+1} and b = F_n. Then we will show that $F_{n+1}=F_n$ is a best approximation. Suppose that a=b is a best approximation. We can assume that a=b 1 = $F_2=F_1$ as otherwise

$$\frac{k\ \ k}{1} = \quad \frac{F_1}{F_0} = j \quad 1j < \quad \frac{a}{b} = \frac{kb\ \ k}{b}$$

contradicting that a=b is a best approximation as 1 < b yet $\frac{}{pn}$ F_{n} pn1

$$\frac{k\ \ k}{1} < \frac{kb\ \ k}{b}:$$

Similarly we can rule out that a=b > 2 = $F_3=F_2$. Thus we can assume that there is an > 2 such that a=b is between $F_n=F_{n\ 1}$ and $F_{n+2}=F_{n+1}$ using Theorem 5. Notice

$$\frac{1}{bF_{n\ 1}} \quad \frac{a}{b} \quad _1 <=314^{a}{}^{+1}$$

k

Proof.

$$\frac{k}{} = kk(\quad 1)k = kk \quad kk = kk \; k$$

2.4. Fibonacci Facts. In this section, we state some facts about Fibonacci numbers that will play an important role in this paper.

Theorem 7. Let $p; q$ 2 N with $p > q$. Then

$$F_pF_{q+1} \quad F_{p+1}F_q = (\quad 1)^pF_{p \quad q}:$$

For a proof of Theorem 7 see [5].

Lemma 2. Let n 2 N then

$$[F_n \quad] = F_{n+1}:$$

3. Construction

In this section, we describe the sequence of functions of interest. This construction appears in [1] for general fractions $p=q$ 2 Q but for this paper we restrict our attention to those fractions of the form $F_n=F_{n+1}$. First, we de ne

$$S^1 = f\,e^{2\,i} \quad : \quad 2\ R=Zg \quad C:$$

For n 2 N with $n >$ 2 we de ne $R_n : S^1 ! \quad S^1$ to be

$$R_n \quad e^{2\,i} \qquad n$$

(a) Step 1

(b) Step 2

(c) Step 3

Figure 1. Process for constructing the critical orbit for f_5 map with the construction.

Figure 2. The map f_4 obtained by the construction.

Lemma 3. Let n 2 N, k 2 N_0 with $n > 2$ and $k < F_{n+1}$. Further

Then note

$$\frac{kF_n}{F_{n+1}} \pmod 1 = \frac{r}{F_{n+1}}:$$

This concludes the proof.

We will now state a few observations about the ordered set Z_n for $n\ 2\ N$ with $n > 2$ that follow from Lemma 3 and from d'Ocagne's identity.

Remark 1.

(1) $z^n_0 = q^n_0$
(2) $q^n_1 = z^n_{F_n}$ if n is odd and $q^n_1 = z^n_{F_n\ \ 1}$ if n is even
(3) $z^n_2 = q^n_{F_n\ \ 2}$
(4) If z^n_k is in the counterclockwise arc of S^1 from z^n_0 to z^n_2 then $w^n_k = 1$ otherwise $w^n_k = 0$.

Using these observations we get that $w^n_k = 1$ if and only if

(7) $$0 < kF_n \pmod{F_{n+1}} \quad F_{n\ \ 2}:$$

Similarly $w^n_k = 0$ if and only if

(8) $$F_{n\ \ 2} < kF_n \pmod{F_{n+1}} \quad 1 \pmod{F_{n+1}}:$$

For $n\ 2\ N$ with $n > 2$ and $k\ 2\ N_0$ with $k < F_{n+1}$, let $l_n(k)$ be such that $z^n_k = q^n_{l_n(}$

4. Results

4.1. Main Result. Split w^n in the following way:

$$w^n = {}^n_1\ {}^n_2\ {}^n_3.$$

Where ${}^n_1$ and ${}^n_3$ are of length $F_{n\ \ 1}$ and ${}^n_2$ is of length $F_{n\ \ 2}$. For all $n; j\ 2\ N$ with $j <\ 3$ and $n >\ 2$, if ${}^n_j = e_1; :::; e_s$ where s is either $F_{n\ \ 1}$ or $F_{n\ \ 2}$ de ne $\overline{{}^n_j}$ to be $e_1; \overline{e_2}; :::; e_s$. If the length of ${}^n_j$ is smaller than 2 then $\overline{{}^n_j} = {}^n_j$. Recall if $x\ 2\ f\ 0; 1g\ \overline{x}$ is $1\ \ x$. The main result of this paper is

Theorem 3 (Inductive construction of the kneading invariant). Suppose $n\ 2\ N$ with $n >\ 2$ then

$$w^n = \begin{cases} w^{n\ 1}\ \overline{{}^{n\ 1}_1}\ {}^{n\ 1}_2 & n \text{ odd} \\ w^{n\ 1}\ {}^{n\ 1}_1\ {}^{n\ 1}_2 & n \text{ even} \end{cases}$$

To prove Theorem 3 we will prove this equivalent lemma.

Lemma 5. For all $n >\ 2$

$$w^n = \begin{cases} {}^{n\ 1}_1\ {}^{n\ 1}_2\ {}^{n\ 1}_1\ \overline{{}^{n\ 1}_1}\ {}^{n\ 1}_2 & n \text{ odd} \\ {}^{n\ 1}_1\ {}^{n\ 1}_2\ \overline{{}^{n\ 1}_1}\ {}^{n\ 1}_1\ {}^{n\ 1}_2 & n \text{ even} \end{cases}$$

Proof. (Lemma 3 implies Theorem 3)
We will use induction on n. See Figure 3 for the base case. Now suppose $n >\ 2$ and

$$w^n = \begin{cases} w^{n\ 1}\ \overline{{}^{n\ 1}_1}\ {}^{n\ 1}_2 & n \text{ odd} \\ w^{n\ 1}\ {}^{n\ 1}_1\ {}^{n\ 1}_2 & n \text{ even} \end{cases}. \tag{9}$$

By Lemma 5 we have

$$w^{n+1} = \begin{cases} {}^n_1\ {}^n_2\ {}^n_1\ \overline{{}^n_1}\ {}^n_2 & n \text{ even} \\ {}^n_1\ {}^n_2\ \overline{{}^n_1}\ {}^n_1\ {}^n_2 & n \text{ odd}. \end{cases}$$

Suppose that n is even. Then by (9) we know ${}^n_3 = {}^{n\ 1}_1\ {}^{n\ 1}_2 = {}^n_1$. If n is odd, we have that ${}^n_3 = \overline{{}^{n\ 1}_1}\ {}^{n\ 1}_2 = \overline{{}^n_1}$. Hence

$$w^{n+1} = \begin{cases} w^n\ \overline{{}^n_1}\ {}^n_2 & n \text{ odd} \\ w^n\ {}^n_1\ {}^n_2 & n \text{ even} \end{cases}.$$

This concludes the proof.

4.2. Examples. We will walk through how to use this pattern. Looking at Figure 3 we see that

$$w^3 = 010$$

and

$${}^3_1 = 0 \qquad {}^3_2 = 1 \qquad {}^3_3 = 0.$$

Using Lemma 3 we see that

$$w^4 = {}^3_1\ {}^3_2\ \overline{{}^3_1}\ {}^3_1\ {}^3_2$$
$$= 01001.$$

Now we have that

$${}^4_1 = 01 \qquad {}^4_2 = 0 \qquad {}^4_3 = 01.$$

This time by Theorem 3,

$$w^5 = w^4\ \overline{{}^4_1}\ {}^4_2$$
$$= 01001000$$

occurs then for all subsequent n', q'_n will be a constant. We formalize this with the following lemma. Recall that for $x \in \mathbb{R}$

$$\|x\| = \min\{|x - n| : n \in \mathbb{Z}\} = \min\{x - \lfloor x \rfloor, \lceil x \rceil - x\}.$$

Lemma 7. For each $k \in \mathbb{N}$ there is a $N_k \in \mathbb{N}_0$ such that for all $n \geq N_k$,

$$\left\| \frac{kF_n}{F_{n+1}} \right\| = \left\| \frac{k}{\phi} \right\|$$

and

$$N_k = \left\lceil \frac{\log_\phi \left(\frac{\phi^2+1}{\phi\,\delta_k} \right) + 1}{2} \right\rceil.$$

Proof. Fix $k \in \mathbb{N}$, set

$$\delta_k = \left\| \frac{k}{\phi} \right\|.$$

Clearly $\delta_k > 0$ and so we know there is N_k such that for all $n \geq N_k$,

$$\left| \frac{kF_n}{F_{n+1}} - \frac{k}{\phi} \right| < \delta_k. \tag{10}$$

Fix $n \geq N_k$. Note that

$$\frac{kF_n}{F_{n+1}} \in \left(\frac{k}{\phi} - \delta_k, \frac{k}{\phi} + \delta_k \right)$$

and that

$$\delta_k = \min \left\| \frac{k}{\phi} \right\|$$

We get

$$\frac{\ ^{n}\ (\ )^{n}}{\ ^{n+1}\ (\ )^{n\ 1}}\ \frac{1}{\ } = \frac{\ ^{n+1}\ (\ )^{n}\ \ ^{n+1} + (\ )^{n\ 1}}{\ ^{n+2}\ (\ )^{n\ 1}}$$

$$= \frac{(\ )^{n} + (\ )^{n\ 1}}{\ ^{n+2}\ (\ )^{n\ 1}}$$

$$= \frac{(\ )^{n+1} + (\ )^{n\ 1}}{\ ^{n+2}\ (\ )^{n\ 1}}$$

$$=(\ ^{2}+1)\ \frac{(\ )^{n\ 1}}{\ ^{n+2}\ (\ )^{n\ 1}}$$

$$= \frac{\ ^{2}+1}{\ }\ \frac{1}{(\ 1)^{n+1}\ \ ^{2n+2}\ 1}$$

$$\frac{\ ^{2}+1}{\ }\ \frac{1}{\ ^{2n+2}\ 1}:$$

Notice

$$\frac{\ ^{2}+1}{\ }\ \frac{1}{\ ^{2n+2}\ 1} < \frac{\ _{k}}{k}$$

happens when

$$n\ \ \frac{1}{2}\log\ \ \frac{\ ^{2}+1}{\overline{k}\ \ _{k}} + 1\ !\ \ 1 = N_k:$$

This proves the claim.

Recall again that for this paper x (mod y) is de ned as the remainder of x divided by y. Lemma 7 allows us to prove the following lemma.

Lemma 8. For each k 2 N if N_k is as de ned above, for all n 2 N with n $N_k + 2$ we have

$$kF_n \pmod{F_{n+1}} = kF_{n\ 1} \pmod{F_n} + kF_{n\ 2} \pmod{F_{n\ 1}}:$$

Proof. Fix k 2 N and let n 2 N such that n $N_k + 2$. Write

$$kF_n = q_n F_{n+1} + r_n$$
$$kF_{n\ 1} = q_{n\ 1} F_n + r_{n\ 1}$$
$$kF_{n\ 2} = q_{n\ 2} F_{n\ 1} + r_{n\ 2}$$

where 0 $r_i < F_{i+1}$ for all i. Note that

$$q_n = \ \frac{kF_n}{F_{n+1}}\ :$$

By Lemma 7, as $n > N_k + 2$ q_n, $q_{n\ 1}$, and $q_{n\ 2}$ will all be equal. Call their common value q. Rewriting the above we see that

$$r_n = kF_n\ \ qF_{n+1}$$
$$r_{n\ 1} = kF_{n\ 1}\ \ qF_n$$
$$r_{n\ 2} = kF_{n\ 2}\ \ qF_{n\ 1}$$

6Td [(=)]TJ/F50 10.9091 Tf 11.515 0 Td [(q)]TJ/F51 7.9701 Tf 4.87 -1.637 Td [(n)]TJ/F53 7.9701 Tf 5.138 e
10.9091 Tf 10.909 0 Td 510n Tf 13.058 -1.636 Td [(n)]TJ/F30 10.9091 Tf 8.667 1.636 Td [(74 -14.557 836(F)]][(
n

Lastly note that for all i in $\{n-2, n-1, n\}$ we have that

$$r_i = kF_i \pmod{F_{i+1}}.$$

And so we have

$$kF_n \pmod{F_{n+1}} = kF_{n-1} \pmod{F_n} + kF_{n-2} \pmod{F_{n-1}}$$

as desired.

To prove Lemma 6 we must prove that for $n \in \mathbb{N}$ with $n > 2$ and $k \in \mathbb{N}_0$ with $k < F_{n-1}$

$$w_k^n = w_k^{n-1}.$$

We need Lemma 8 to apply for each $n \in \mathbb{N}$ such that $k < F_{n-1}$

We can use the fact that $[F_n = \ ] = F_{n \ 1}$ to see that this is the same as

$$\frac{1}{} \quad \frac{F_{m \ 1}}{F_m} \qquad \frac{^2 + 1}{(\ ^{2m+2} \quad 1)}:$$

Expanding this out we get

$$\frac{^m \quad (\quad) \ ^m \qquad ^m + \quad (\quad) \ ^{m+1}}{(\ ^m \quad (\quad) \ ^m)} \qquad \frac{^2 + 1}{(\ ^{2m+2} \quad 1)}$$

$$\frac{(\quad) \ ^m + \quad (\quad) \ ^{m+1}}{(\ ^m \quad (\quad) \ ^m)} \qquad \frac{^2 + 1}{(\ ^{2m+2} \quad 1)}$$

$$\frac{^2 + 1}{j \ ^{m+1}(\ ^m \quad (\quad) \ ^m)j} \qquad \frac{^2 + 1}{(\ ^{2m+2} \quad 1)}$$

$$^{m+1}(\ ^m \quad (\quad) \ ^m) \qquad (\ ^{2m+2} \quad 1)$$

$$(\ ^{2m} + (\quad 1)^{m \ 1}) \qquad (\ ^{2m+2} \quad 1)$$

$$^{2m} + 1 \qquad ^{2m+2} \quad 1$$

$$2 \qquad ^{2m+1}:$$

Which is true for all $m \quad 1$.

We are now ready to prove Lemma 6.

Lemma 6. Let $n \ 2 \ N$ with $n > 2$,

$$^n_1 = \quad ^{n \ 1}_1 \quad ^{n \ 2}_2$$

Proof. Recall

$$^n_1 = w^n_1 ::: w^n_{F_{n \ 1}}$$
$$^{n \ 1}_1 \quad ^{n \ 1}_2 = w^{n \ 1}_1 ::: w^{n \ 1}_{F_{n \ 1}}$$

Thus we must prove for all $n; k \ 2 \ N_0$ with $n > 2$ and $k < F_{n \ 1}$,

$$w^n_k = w^{n \ 1}_k:$$

We will proceed by induction on n. The case $n = 3$ can be veri ed directly from Figure 3. Fix n and suppose that for $k \ 2 \ N_0$ with $k < F_n$

$$w^n_k = w^{n \ 1}_k:$$

We consider two cases: $w^n_k = 1$ and $w^n_k = 0$. First we suppose that $w^n_k = w^{n \ 1}_k = 1$. Using 7 we get

$$0 < kF_{n \ 1} \pmod{F_n} \quad F_{n \ 2}$$

and

$$0 < kF_{n \ 2} \pmod{F_{n \ 1}} \quad F_{n \ 3}:$$

Summing and applying Lemma 8 we get:

$$0 < kF_n \pmod{F_{n+1}} \quad F_{n \ 1}:$$

Hence $w^n_k = 1$ by 7.

Now suppose $w^{n \ 1}_k = 0$. Thus

$$F_{n \ 2} < kF_{n \ 1} \pmod{F_n}$$

and

$$F_{n \ 3} < kF_{n \ 2} \pmod{F_{n \ 1}}$$

Summing and applying Lemma 8 we get

$$F_{n \ 1} < kF_n \pmod{F_{n+1}}$$

and so $w^n_k = 0$. In either case $w^n_k = w^{n-1}_k$ concluding the proof.

4.4. Proof of Lemma 13.

Lemma 13. Let $n \in \mathbb{N}$ with $n > 2$ then

$$\kappa^n_3 = \begin{cases} \overline{\kappa^{n-1}_1 \kappa^{n-1}_2} & n \text{ odd} \\ \kappa^{n-1}_1 \kappa^{n-1}_2 & n \text{ even} \end{cases}$$

Proof. Fix $n \in \mathbb{N}$ with $n > 2$. By Lemma 6 this is the same as proving that for all k with $F_n < k \leq F_{n+1}$ with $k \neq F_n + 2$

$$w^n_k = w^n_{k-F_n}$$

while

$$w^n_{F_{n+2}} = \begin{cases} w^n_2 & n \text{ odd} \\ \overline{w^n_2} & n \text{ even.} \end{cases}$$

Fix $k \in \mathbb{N}$ with $F_n < k < F_{n+1}$, (F_{n+1} is handled later). Set $p = k - F_n$ and note $p \leq F_{n-1}$. Using d'Ocagne's identity, compute

$$\begin{aligned}(p + F_n)F_n \pmod{F_{n+1}} &= (pF_n + F_n^2) \pmod{F_{n+1}} \\ &= (pF_n + (-1)^{n-1}) \pmod{F_{n+1}}:\end{aligned}$$

We now consider the two cases: n is even and n is odd. Suppose first that n is even. We wish to prove that $w^n_k = w^n_p$.

Suppose that $w^n_p = 1$ then we know that

$$0 < pF_n \pmod{F_{n+1}} \leq F_{n-2}$$

by (7). Adding $-1 \pmod{F_{n+1}}$ gives us

$$-1 \pmod{F_{n+1}} < kF_n \pmod{F_{n+1}} \leq F_{n-2} \tag{12}$$

In addition notice we assumed $k < F_{n+1}$ and so by Lemma 4 we know that $kF_n \pmod{F_{n+1}} \neq 0$ as $F_{n+1}F_n \pmod{F_{n+1}} = 0$. Thus we can rewrite 12 as

$$0 < kF_n \pmod{F_{n+1}} \leq F_{n-2} \tag{13}$$

and conclude $w^n_k = 1 = w^n_p$ as desired. Now assume that $w^n_p = 0$.

$$F_{n-2} < pF_n \pmod{F_{n+1}} \leq -1 \pmod{F_{n+1}}:$$

Adding $-1 \pmod{F_{n+1}}$,

$$F_{n-2} \leq kF_n \pmod{F_{n+1}} \leq -2 \pmod{F_{n+1}} \leq -1 \pmod{F_{n+1}}:$$

Again note that $k > F_n$ and so we know that $kF_n \pmod{F_{n+1}} \neq F_{n-2}$ as $2F_n \pmod{F_{n+1}} = F_{n-2}$. Thus

$$F_{n-2} < kF_n \pmod{F_{n+1}} \leq -1 \pmod{F_{n+1}}:$$

as $w^n_k = 0 = w^n_p$.

Now assume that n is odd. We repeat the same process. Suppose that $w^n_p = 1$. Then

$$0 < pF_n \pmod{F_{n+1}} \leq F_{n-2}$$

by (7). Adding $1 \pmod{F_{n+1}}$ gives us

$$0 < kF_n \pmod{F_{n+1}} \leq F_{n-2} + 1: \tag{14}$$

Note that if $k = F_n + 2$ then $p = 2$ and so $pF_n \pmod{F_{n+1}} = F_{n-2}$ and $kF_n \pmod{F_{n+1}} = F_{n-2} + 1$. Hence if $k \neq F_n + 2$ then

$$0 < kF_n \pmod{F_{n+1}} \leq F_{n-2}$$

and $w_k^n = 1 = w_p^n$. If $k = F_n + 2$ then

$$kF_n \quad (\mathrm{mod}\ F_{n+1}) = F_{n\ \ 2} + 1 > F_{n\ \ 2}$$

and so $w_k^n = 0 = \overline{w_p^n}$ as desired.

Lastly suppose $w_p^n = 0$. Then

$$F_{n\ \ 2} < pF_n \quad (\mathrm{mod}\ F_{n+1}) \qquad 1 \quad (\mathrm{mod}\ F_{n+1}):$$

Adding 1 (mod F_{n+1}),

$$F_{n\ \ 2} < F_{n\ \ 2} + 1 < kF_n \quad (\mathrm{mod}\ F_{n+1}) \ +1$$