A PHASE SPACE TRANSFORM ADAPTED TO THE WAVE EQUATION

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Abstract. Wave packets emerged in recent years as a very useful tool in the study of nonlinear wave equations. In this article we introduce a phase space transform adapted to the geometry of wave packets, and use it to characterize and study the associated classes of pseudodifferential and Fourier integral operators.

1. Introduction

A natural way to study pseudodifferential and Fourier integral operators is by means of phase space transforms. This is easiest to understand within the framework of the S^0_{00} calculus, where the localization occurs on the unit scale both in position and in the frequency. The corresponding phase space transform is precisely the Bargmann transform,

$$Tu(x,\xi) = c_n \int_{\mathbb{R}^n} e^{i\xi(x-y)} e^{-\frac{(x-y)^2}{2}} u(y)dy$$

The Bargmann transform is an isometry from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{C}^n)$ so an inverse for it is provided by the adjoint operator. This inverse is not uniquely determined since T is not onto. Precisely, the range of T consists of those functions satisfying a Cauchy-Riemann type equation, $i\partial_\xi v = (\partial_x - i\xi)v$. The connection with the S^0_{00} type calculus is provided by the following simple result:

Theorem 1.1. *([14]) Let $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a linear operator. Then $A \in OPS^0_{00}$ if and only if the kernel K of S*

2.

This shows that within R_ξ we can freeze the metric to g_ξ and obtain an equivalent distance. Thus R_ξ roughly coincides with a unit sized ball with respect to the metric g.

The dual balls to R_ξ in the physical space are denoted by

$$R_\xi^{-1} = \{y \quad \mathrm{R}^n;\ g_\xi^{-1}(y) \leq 4\}$$

These are roughly boxes of size $\xi^{\ -1}$ in the ξ direction and $\xi^{\ -\frac{1}{2}}$ in all directions normal to ξ. Then we can describe small balls in the phase space roughly by

$$B((x,\xi),\ ) \approx (x + \ R_\xi^{-1}) \times \ R_\xi, \qquad \leq 1$$

To measure the regularity of functions on the scale given by the metric g we use the notation

$$| \ {}^k a(x,\xi)|_g = \sup\{(\Pi_{j=1}^k v_j \quad)a(x,\xi);\ g_{x,\xi}(v_j) \leq 1,\ j = \overline{1,k}\}$$

In the rescaled setting we need to prove that the derivatives of the components of g_r at $\zeta = e_1$ are uniformly bounded with respect to $r \geq 1$. But this is straightforward. □

The above results show that the short range geometry is essentially flat, and is described by the frozen metric g_ξ. This is no longer the case for the long range geometry. To characterize it we begin with a simpler result allowing us to compare g at di□erent points:

Lemma 2.3. *Let $\xi, \eta \quad \mathrm{R}^n$. Then*

$$g_\xi^{-1}(u) \square g_\eta^{-1}(u)(1 + g_\eta(\xi - \eta))$$

and the dual bound

$$g_\xi(u) \square g_\eta(u)(1 + g_\xi(\xi - \eta))$$

Proof. Since $g_\eta(\eta) \leq 1$ it follows that

$$(1 + g_\eta(\xi - \eta)) \approx 1 + g_\eta(\xi)$$

We write ξ as

$$\xi = \frac{\xi \cdot \eta}{\eta^2}\eta + \frac{1}{\eta^2}(\xi \quad \eta)\eta$$

Then we express the component of u along ξ as

$$\xi \cdot u = \frac{\xi \cdot \eta}{\eta^2}\eta \cdot u + \frac{1}{\eta^2}(\xi \quad \eta)\eta \cdot u$$

Hence we have

$$g_\xi^{-1}(u) \square \frac{\xi^{\ \ 2}}{\xi^2\eta^4}(|\xi \cdot \eta|^2|\eta \cdot u|^2 + |\xi \quad \eta|^2|u|^2|\eta|^2) + \ \ \xi \ \ |u|^2$$

$$\square$$

$$\square g_\eta^{-1}(u) \qquad |^2|$$

Proof. We first note that $g_{\xi_1}(\eta - \xi_1) \approx 1$, so without any restriction in generality we can take $\eta = \xi_1$. By Lemma 2.3 we also have

$$g_{\xi_1}(\xi - \xi_2) \square\, g_{\xi_2}(\xi - \xi_1)(1 + g_{\xi_1}(\xi_1 - \xi_2)) \square\, (1 + g_{\xi_1}(\xi_1 - \xi_2))$$

Then the "□" bound follows. The opposite inequality must also be true by symmetry.

□

Finally, we give a complete characterization of the distance d.

Theorem 2.5. *We have* □

$$1 + d(\xi, \eta) \approx \ln \; g_\xi(\xi - \eta) + \frac{\xi}{\eta} + \frac{\eta}{\xi} + \; \xi, \eta \;_{-} \tag{1}$$

This reduces the proof of (2) to the case when $\xi = \eta$. We proceed as in the first part. First we construct a geodesic whose length is comparable to $\ln(1 + g_\xi^{-1}(x-y)) + |x-y|$.

We consider the piecewise straight trajectory

$$\gamma(\zeta) : (x, \xi) \to (x, \zeta) \to (y, \zeta) \to (y, \xi)$$

where $\zeta \in [0, \xi]$. This has length

$$l(\gamma(\zeta)) \approx \sqrt{g_\zeta^{-1}(x-y)} + \ln \frac{\xi}{\zeta} = f(\zeta)$$

Then we optimize its length with respect to $\zeta \in [0, \xi]$ and show that

$$\min_{\lambda \in [0,1]} f(\lambda) \approx \ln(1 + g_\xi^{-1}(x-y)) + |x-y|$$

For one direction we note the trivial bound $f(\lambda) \geq |x-y|$. In addition,

$$f(\lambda) \gtrsim \ln(1 + g_\zeta^{-1}(x-y)) + \ln \frac{\xi}{\zeta} \gtrsim \ln(1 + g_\xi^{-1}(x-y))$$

For the other direction we consider three cases:

(i) $g_\xi^{-1}(x-y) \leq 1$. Then we set $\zeta = \xi$ and

$$f(\zeta) \approx \sqrt{g_\xi^{-1}(x-y)} \approx \ln(1 + g_\xi^{-1}(x-y))$$

(ii) $g_\xi^{-1}(x-y) > 1$ and $|x-y| < 1$. Then we choose ζ so that $g_\zeta^{-1}(x-y)$

Taking advantage of the analysis in the proof of (1) it follows that

$$I(y) \square \square\overline{g_\zeta^{-1}(x-y)} + d(\xi,\zeta) \approx \square\overline{g_\zeta^{-1}(x-y)} + \ln\frac{\xi}{\zeta} = f(\zeta)$$

which concludes the proof.

□

Later in the paper we want to use the distance d to express phase space kernel bounds of pseudodi□erential operators. For this purpose the last term in (2) proves inconvenient. Hence we introduce a second distance $\tilde{d}$ which is a technical modification of the distance d. The term $|x-y|$ arises when the geodesic moves between x and y

Proof. By (6) we have

$$\int_{\mathbb{R}^{2n}} e^{-N\tilde{d}_{even}((x,\xi),(y,\eta))} dy d\eta \lesssim \int_{\mathbb{R}^{2n}} (1+g_\xi(\xi-\eta)+g_\xi^{-1}(x-y))^{-cN} dy d\eta \lesssim 1$$

□

3. The phase space transform

The coherent states adapted to the phase space structure induced by the metric g are bump functions $\varphi_{x,\xi}$ which are localized in frequency within R_ξ and in position in $x + R_\xi^{-1}$. We certainly cannot have sharp localization at both ends. Unlike in the case of the classical Bargmann transform, we choose to have sharp frequency localization in order to prevent potentially troublesome concentrations at the origin.

Definition 3.1. *Let h be an L^2 normalized smooth function supported in $B(0, \frac{1}{16})$. Then the coherent states $\varphi_{x,\xi}$ are defined by*

$$\hat{\varphi}_{x,\xi}(\eta) = \langle\eta\rangle^{-\frac{n+1}{4}} h(g_\eta(\xi - \eta)) e^{-i\eta x} \tag{7}$$

We also introduce the notation

$$\varphi_\xi(\eta) = \langle\eta\rangle^{-\frac{n+1}{4}} h(g_\eta(\xi - \eta))$$

We note that $\varphi_{x,\xi}(y)$ are not exactly L^2 normalized but

$$\|\varphi_{x,\xi}\|_{L^2} \approx 1$$

One can also see that they can be represented in the form

$$\varphi_{x,\xi}(y) = \langle\xi\rangle^{\frac{n+1}{4}} k_\xi(y - x) e^{i\xi(y-x)} \tag{8}$$

where k_ξ is a smooth bump function on the R_ξ^{-1} scale.

Lemma 2.1 allows us to describe the frequency support of the coherent states:

Lemma 3.2. *For $\xi, \eta \in \mathbb{R}^n$ we have*

$$\hat{\varphi}_{x,\xi}(\eta) \neq 0 \Longrightarrow \xi \in R_\eta, \eta \in R_\xi$$

Now we can define our phase space transform:

Definition 3.3. *The phase transform T is*

$$Tu(x, \xi) = \int_{\mathbb{R}^n} u(y)\overline{\varphi_{x,\xi}(y)}\, dy, \qquad u \in S(\mathbb{R}^n) \tag{9}$$

We can also express Tu in terms of the Fourier transform of u,

$$Tu(x,\xi) = \int_{\mathbb{R}^n} \hat{u}(\eta)\overline{\hat{\varphi}_{x,\xi}(\eta)}\, d\eta$$

The adjoint operator T^* is given by

$$T^* f(y) = \int_{\mathbb{R}^n} f(x,\xi)\varphi_{x,\xi}(y)\, dx d\xi$$

It is easy to see that

Proposition 3.4. *The following mapping properties hold:*

$$T : S(\mathrm{R}^n) \to S(\mathrm{R}^{2n}), \qquad T^* : S(\mathrm{R}^{2n}) \to S(\mathrm{R}^n)$$

By duality this allows us to extend T and T^* to linear operators

$$T : S'(\mathrm{R}^n) \to S'(\mathrm{R}^{2n}), \qquad T^* : S'(\mathrm{R}^{2n}) \to S'(\mathrm{R}^n)$$

As for the classical Bargmann transform we have

Proposition 3.5. *The phase space transform T is an isometry*

$$T : L^2(\mathrm{R}^n) \longrightarrow L^2(\mathrm{R}^{2n}).$$

Thus we have the inversion formula

$$u(y) = \int_{\mathbb{R}^{2n}} Tu(x,\xi)\varphi_{x,\xi}(y)\, dx d\xi \tag{10}$$

Proof. A straightforward calculation using the Fourier inversion formula leads to:

$$\begin{aligned}
\widehat{T^*Tu}(\zeta) &= \int \hat{u}(\eta)\overline{\hat{\varphi}_{x,\xi}(\eta)}\hat{\varphi}_{x,\xi}(\zeta)\, d\eta dx d\xi \\
&= \int \hat{u}(\eta)e^{ix(\zeta-\eta)}h(g_\zeta(\xi-\zeta))\overline{h(g_\eta(\xi-\eta))}\, \langle\eta\rangle^{-\frac{n+1}{4}} \langle\zeta\rangle^{-\frac{n+1}{4}}\, d\eta dx d\xi \\
&= \langle\zeta\rangle^{-\frac{n+1}{2}}\hat{u}(\zeta) \int |h(g_\zeta(\xi-\zeta)|^2 d\xi \\
&= \hat{u}(\zeta)
\end{aligned}$$

□

4. Symbol classes and pseudodifferential operators

As a starting point we define the symbol class which is associated to the metric g. Precisely, we consider symbols which are smooth in ξ on the R_ξ scale and in x on the R_ξ^{-1} scale. To describe their size we need

Definition 4.1. *A function $m : \mathbb{R}^{2n} \to \mathbb{R}^+$ is slowly varying function with respect to the metric g if*

$$d((x,\xi),(y,\eta)) \leq 1 \quad \Longrightarrow \quad m(x,\xi) \approx m(y,\eta)$$

and there exists k so that

$$m(x,0) \lesssim m(y,0)(1+|x-y|)^k$$

Obvious examples of slowly varying functions are 1, $\langle\xi\rangle^s$. Since within each unit ball m can change by a fixed factor, we easily obtain

$$m(x,\xi) \lesssim m(y,\eta) e^{C\tilde{d}((x,\xi),(y,\eta))} \tag{11}$$

Definition 4.2. *Let $m : \mathbb{R}^{2n} \to \mathbb{R}^+$ be a slowly varying function with respect to the metric g. The symbol $a \in C^\infty(\mathbb{R}^{2n})$ belongs to $S(m,g)$ if it satisfies the following estimates:*

$$|\partial^\alpha a(x,\xi))|_g \leq c_\alpha m(x,\xi) \tag{12}$$

If $m = 1$ then we simply write $S(g)$.

Unfortunately, the corresponding class of operators $OPS(g)$ is not so well behaved because a certain degree of concentration at frequency 0 is permitted for the corresponding pseudodifferential operator. This phenomena is similar to what happens in the case of $S^0_{1,1}$ symbols, only it gets worse here. To remedy this we consider the analogue of Hörmander's [8] $\tilde{S}^0_{1,1}$ class (see also [1], [4], [9]):

Definition 4.3. *Let $m : \mathbb{R}^{2n} \to \mathbb{R}^+$ be an even slowly varying function with respect to the metric g. The symbol $a \in S(m,g)$ belongs to $\tilde{S}(m,g)$ if in addition it satisfies the following estimates:*

$$|S_{<\lambda}(D_x+\xi)a(x,\xi)| \leq c_N m(x,\xi) \tag{13}$$

Our main result is to characterize the symbols in the $\tilde{S}(m, g)$ class in terms of the phase space transform.

Theorem 4.5. *Let $A : S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ be a linear operator. Let $m : \mathbb{R}^{2n} \to \mathbb{R}^+$ be an even slowly varying function with respect to the metric g. Then $A \quad OP\tilde{S}(m, g)$ if and only if the kernel K of TAT^* satisfies the following bounds:*

$$|K(x_1, \xi_1, x_2, \xi_2)| \le c_N m(x_1, \xi_1) e^{-N\tilde{d}_{even}} \tag{14}$$

Since

$$\frac{1}{1 + g_{\xi_2}(\xi_1 - \xi_2)} \square \frac{\xi_2^{\ 2}}{\xi_1^{\ 2}}$$

the bound (16) is as strong as (14) in the region $\{|\xi_2| \square |\xi_1|\}$. This

Then we can simply apply the bound (23) provided we know the stronger symbol estimate

$$|\,{}^{k}S_{<\lambda}(D_x + \xi)a(x,\xi)|_g \lesssim c_{k,N}\, m(x,\xi)(\lambda/\ \xi\ )^N \tag{25}$$

For k

where due to the "min" factor we are allowed to freely interchange g_{ξ_1} and g_{ξ_2}. We integrate successively with respect to x_2 and then x_1 to obtain

$$|a(x,\xi)| \lesssim \int \frac{1_{R_{\xi_2}}(\xi) \min\{ \xi_1 / \xi_2 , \ \xi_2 / \xi_1 \}^N m(x,\xi_2)}{|R_{\xi_1}|^{\frac{3}{2}} |R_{\xi_2}|^{\frac{1}{2}} (1 + g_{\xi_2}(\xi_2 - \xi_1))^N} d\xi_i$$

Since

$$|R_{\xi_1}|^{-\frac{3}{2}} |R_{\xi_2}|^{-\frac{1}{2}} \min\{ \xi_1 / \xi_2 , \ \xi_2 / \xi_1 \}^N \lesssim |R_{\xi_2}|^{-2}$$

we also integrate in ξ_1 to obtain

$$|a(x,\xi)| \lesssim m(x,\xi) \int 1_{R_{\xi_2}}(\xi) |R_{\xi_2}|^{-1} d\xi_2 \lesssim m(x,\xi)$$

b2) The x derivatives of a.

We use this and continue as in (b1). The additional factor

$$(1 + g_\xi^{-1}(x_2 - x))^k$$

is again negligible due to the off diagonal decay of K, so we obtain

$$|(1 - \Delta_{g_\xi})^k a(x, \xi)| \lesssim m(x, \xi) \tag{28}$$

b4) Additional decay near frequency ξ. From (26) we obtain

$$S_{<\lambda}(D_x + \xi)a(x, \xi) = \int e^{-ix\cdot(\xi-\eta)} K(x_1, \xi_1, x$$

which yields

$$\sup_{y,\eta} \int |K(y, \eta, x, \xi)| dx d\xi < \infty$$

For this we seek to propagate the norm

$$g_{\xi^t}^{-1}(y) + g_{\xi^t}(\eta)$$

via a Gronwall type inequality. We consider the two components in each of the two terms. By (iii) we have

$$|a_{\xi x}(t, x^t, \xi^t)y| \lesssim \xi^t{}^{-\frac{1}{2}} g_{\xi^t}^{-1}(y)^{\frac{1}{2}}, \qquad |a_{\xi\xi}(t, x^t, \xi^t)\eta| \lesssim \xi^t{}^{-\frac{1}{2}} g_{\xi^t}(\eta)^{\frac{1}{2}}$$

hence the Euclidean length of y is easy to propagate. We need a more refined computation for the component in the ξ direction:

$$\frac{d}{dt}(y\xi^t) = \xi^t a_{\xi x} \quad {}^t a \ldots$$

Next we compute the action of pseudodifferential operators on coherent states.

Lemma 6.4. *We have*

$$A(t,x,D)\varphi_{y,\eta} = (a(t,y,\eta) - i(a_x(t,y,\eta)\partial_\eta - a_\xi(t,y,\eta)(\partial_y + i\eta))\,\varphi_{y,\eta} + r_{y,\eta}$$

where the remainder r satisfies

$$\langle r_{y,\eta}, \varphi_{z,\zeta}\rangle = O(e^{-N d((y,\eta),(z,\zeta))})$$

Proof. The variable t is irrelevant here so we drop it. We expand a in a modified Taylor series around (y,η),

$$a(x,\xi) = a(y,\eta) + a_\xi(y,\eta)(\xi-\eta) + a_x(y,\xi)(x-y) + b(x,\xi)$$

At the operator level this becomes

$$A(x,D) = a(y,\eta) + a_\xi(y,\eta)(D_x - \eta) + (x-y)a_x(y,D) + b(x,D) \tag{30}$$

We apply this to $\varphi_{y,\eta}$ and use the expression (7) to evaluate each of the terms. For the first term we need to do nothing. The Fourier transform of the second term is

$$F(a_\xi(y,\eta)(D_x-\eta)\varphi_{y,\eta}) = ia_\xi(y,\eta)(\partial_y + i\eta)\hat{\varphi}_{y,\eta}(\xi)$$

For the third term we compute

$$\begin{aligned}
F((x-y)a_x(y,D)\varphi_{y,\eta}) = &-(D_\xi + y)[a_x(y,\xi)\hat{\varphi}_{y,\eta}(\xi)] \\
&= e^{-iy\xi}a_x(y,\xi)D_\xi\varphi_\eta(\xi) + ia_{x\xi}(y,\xi)\hat{\varphi}_{y,\eta}(\xi) \\
&= e^{-iy\xi}a_x(y,\eta)D_\eta\varphi_\eta(\xi) \\
&+ e^{-iy\xi}(\langle\xi\rangle^{-1}a_x(y,\xi) - \langle\eta\rangle^{-1}a_x(y,\eta))\langle\xi\rangle D_\xi\varphi_\eta(\xi) \\
&+ e^{-iy\xi}\langle\eta\rangle^{-1}a_x(y,\eta)(\langle\xi\rangle D_\xi - \langle\eta\rangle D_\eta)\psi_\eta(\xi) \\
&+ ia_{x\xi}(y,\xi)\hat{\varphi}_{y,\eta}(\xi)
\end{aligned}$$

The first term is what we want, everything else must go into r. The second term is controlled due to the almost homogeneity of $a_x(x,\xi)$ in (ii). For the third term we use Lemma 6.3. For the fourth we use (iii).

Finally we consider the last term in (30). We claim that the symbol B satisfies

$$b \in \tilde{\tilde{S}}(m,g), \qquad m = e^{N_0\tilde{d}(x,\xi;y,\eta)}$$

By Theorem 4.14 this shows that its contribution can be included in $r_{y,\eta}$. Derivatives of b of order two and higher can be estimated directly using (ii) and (iii). The conditions (13) and (17) also follow trivially from the similar conditions for a. Hence it suffices to verify that

$$|b(x,\xi)| \lesssim e^{N\tilde{d}(x,\xi;y,\eta)}, \qquad |\nabla_g b(x,\xi)| \lesssim e^{N\tilde{d}(x,\xi;y,\eta)}$$

The first bound follows from the second by integration along a geodesic. For the second we integrate the second derivative along a geodesic. This works well within balls of size one, but since g is changing we loose a fixed factor when we move from a ball to the next one. This gives the above bound with $\tilde{d}$ replaced by d. To account for the change to $\tilde{d}$ we need to consider geodesics on the set $\xi = 0$. There the metric is Euclidean, so integration only gives linear rather than exponential growth.

□

This allows us to conjugate the operator A with respect to T^*. We define the selfadjoint phase space operator

$$\tilde{A} = (a(t, y, \eta) + i(a_x \quad a($$

Corollary 6.6. *We have*

$$A - A^* \quad OP\tilde{\tilde{S}}(1, g)$$

This implies that the evolution (29) is L^2 well-posed, i.e. that the evolution operators $S(t, s)$ are L^2 bounded. Now we show that they are FIO's associated to the canonical transformations $\chi(t, s)$. Without any restriction in generality we take $s = 0$. We need to obtain bounds for the kernel K_t of $TS(t, 0)T^*$. More generally, given a solution $u = S(t, 0)u_0$ to (29) we seek to control the flow for Tu.

By Lemma 6.5 and Corollary 6.6 we can write

$$0 = T(D_t + A(x, D))u = (D_t + \tilde{A})Tu + R_0(t)u$$

where $R_0(t)T^*$ has a kernel with rapid decay off the diagonal.

Hence if u solves (29) then $v = Tu$ solves

$$(D_t + \tilde{A})v = R_1(t)v$$

which is a transport equation modulo a good integral part. We pull this back to time 0 using the $\tilde{A}$ flow $\tilde{S}(t, s)$,

$$w(t) = \tilde{S}(0, t)v(t)$$

This solves

$$D_t w = R_2(t)w, \qquad R_2(t) = \tilde{S}(0, t)R_1(t)\tilde{S}(t, 0)$$

The pull back R_2 of R_1 still has a kernel with rapid off diagonal decay since $\tilde{S}(t, 0)$ corresponds to the transport along the Hamilton flow which is symplectic therefore measure preserving, and it also is $\tilde{d}$ bi-Lipschitz.

Finally, to get bounds for w we take absolute values above

$$\partial_t |w(t, x, \xi)| \leq c_M \int |w(t, y, \eta)| e^{-M\tilde{d}((x,\xi),(y,\eta))} dy d\eta \tag{31}$$

and use the maximum principle for $|w|$.

Precisely, in order to obtain bounds for the kernel of $TS(t, 0)T^*$ we take initial data $u_0 = \varphi_{x_2,\xi_2}$ and we want to prove that

$$|v(t, x_1, \xi_1)| \leq c_N e^{-N\tilde{d}((x_1,\xi_1),\chi(t,0)(x_2,\xi_2))}$$

This translates to

$$|w(t, x, \xi)| \leq c_N e^{-N\tilde{d}((x,\xi),(x_2,\xi_2))}$$

The initial data for w is

$$w(0) = T\varphi_{x_2,\xi_2}$$

therefore satisfies the above inequality with some constant $c_N(0)$. We claim that if C is sufficiently large depending on c_{2N} in (31) then

$$|w(t,x,\xi)| \leq e^{Ct} c_N(0) e^{-N\tilde{d}((x,\xi),(x_2,\xi_2))}$$

For this we verify that the right hand side is a supersolution for (31) with $M = 2N$. We need to verify that

$$Ce^{-N\tilde{d}((x,\xi),(x_2,\xi_2))} \geq C_M \int e^{-2N\tilde{d}((x,\xi),(y,\eta))} e^{-N\tilde{d}((y,\eta),(x_2,\xi_2))} dy d\eta$$

Indeed from the triangle inequality we can bound the right hand side integrand by

$$e^{-N\tilde{d}((x,\xi),(y,\eta))} e^{-N\tilde{d}((x,\xi),(x_2,\xi_2))}$$

and then use Lemma (2.6) to carry out the integration.

□

Let $B(0)$ be a pseudodifferential operator. Conjugating it with respect to the $D_t + A$ flow we obtain a time dependent family of operators

$$B(t) = S(t,0)B(0)S(0,t)$$

Given a slowly varying weight m with respect to the metric g, the composition result in Theorem 5.4 yields

Proposition 6.7. *Assume that $B(0) \in OP\tilde{\tilde{S}}(m,g)$. Then for all t we have $B(t) \in OP\tilde{\tilde{S}}(m \circ \chi(t,0), g)$.*

We would like to obtain an Egorov type result, i.e. to characterize the symbol of $B(t)$ in terms of the symbol of $B(0)$. In the context of the above result this is not possible since the metric g is exactly on the scale ofise result1697[(g)]TD re())]T6metricic

Thus we have

$$c(0) \in \tilde{\tilde{S}}(m_{x_0,\xi_0}, g)$$

By the previous proposition, this yields

$$c(t) \in \tilde{\tilde{S}}(m_{x_0,\xi_0} \circ \chi(t,0), g)$$

Using (12) for $c(t)$ in the unit ball centered at (x_0, ξ_0) we conclude that

$$|b(t, x_0^t, \xi_0^t) - b(0, x_0, \xi_0)| \lesssim m(x_0, \xi_0)$$

and also that $b(t)$ satisfies (12) for $|\alpha| \geq 1$ with respect to the weight $m \circ \chi(0, t)$.

The conditions (13), (17) for $b(t)$ follow directly from the similar conditions for $c(t)$ with $(x, \xi) = (x_0, \xi_0)$. □

7. Half-wave operators and paradifferential calculus

A large class of symbols which satisfy the conditions of the previous section can be obtained from half wave operators whose coefficients are mollified in a paradifferential fashion. Precisely, we begin with a real symbol $a(x, \xi)$ which is homogeneous of order 1 in ξ, and satisfies the following regularity conditions:

(a) $a(x, \xi)$ is smooth in ξ.

(b) $|\partial_x a(x, \xi)| \lesssim |\xi|$

Lemma 7.1. *We have*

$$d_{hom}((x_0,\xi_0),(x_1,\xi_1)) \approx |x_0 - x_1| + |\xi_0 - \xi_1| + |(x_0 - x_1)\xi_0|^{\frac{1}{2}} \tag{32}$$

Proof. We first prove the inequality

$|x$

Proof. Consider $\gamma_0(t) = (x_0(t), \xi_0(t))$ respectively $\gamma_1(t) = (x_1(t), \xi_1(t))$ two trajectories of the Hamilton flow in $[0, T]$. We take > 0 and T small and prove that

$d_{hom}(\gamma_0(0)_{hom}$ $(\gamma_0$ $t)$ T

where η is a unit vector normal to ξ_0. As a function of x the function $\eta(a_x(x,\xi_0(t))$ is Lipschitz in directions perpendicular to ξ_0 and Hölder 1/2 in the ξ_0 direction. This suffices for the desired conclusion. □

One can easily prove that $A - A^*$ is L^2 bounded, which implies that the above evolution is well-posed in L^2. However, due to the low regularity of the coefficients there is considerable interaction between different frequencies. Thus in order to better understand the flow it is convenient to replace the operator A with an L^2 bounded perturbation

Finally we consider the regularity of the derivatives of $\tilde{a}$ of second order and higher,

$$ {}^{\alpha}_{g_\xi^{-1}}\ {}^{\beta}_{g_\xi}\tilde{a}(x,\xi) = {}^{\alpha}_{g_\xi^{-1}}\ {}^{\beta}_{g_\xi}(S_\xi(D_x)a(x,\xi)) $$

We first argue why we gain a $|\xi|^{-1}$ factor. In the case of x derivatives, we can put onto a either one derivative in the ξ direction or two derivatives perpendicular to ξ. In both cases we gain $|\xi|^{-1}$.

In the case of ξ derivatives we consider three cases. If one ξ derivative falls on S_ξ then we obtain a similar symbol $\tilde{S}_\xi$ but which in addition is supported away from the origin. Then we use the bound

$$ |\tilde{S}_\xi(D)a(x)| \ \square\ |\xi|^{-1}(\ |\xi|^{-1}\xi \quad a \ _{L^\infty} + \ |\xi|^{-2}(\xi \qquad)^2 a \ _{L^\infty}) $$

If we have two derivatives perpendicular to ξ on $a(x,\xi)$ then we gain $|\xi|^{-1}$ because of the regularity of a. If we have two derivatives of which at least one is in the ξ direction then we get 0 because of the homogeneity of a.

Consider now the mixed case. If we have one x derivative and one ξ derivative on a both in directions perpendicular to ξ then we gain $|\xi|^{-\frac{1}{2}}$ from each. If instead the ξ derivative is in the ξ direction, this is precisely the one case we do not need.

Additional ξ derivatives which fall on a are well behaved since a is smooth and homogeneous in ξ. The ones which fall on S_ξ on the other hand yield symbols of similar size and support. Additional x derivatives have a similar effect on the symbol of S_ξ.

□

Proof of Proposition 7.4. We introduce an intermediate weaker symbol regularization,

$$ \tilde{a}_0(x,\xi) = S_{\langle\ x\rangle}a(x,\xi) $$

then we

where B is the bilinear multiplier with symbol

$$b(\eta, \xi) = 1 - h(\frac{\eta}{4\ \xi})$$

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