Topology Prelim Fall 2021.

December 12, 2021

There are 5 questions in this exam each consisting of 3 subquestions. The best 9 out of 15 subquestions will be used as your course nal score for MATH440. If you are a graduate student then this nal also counts as a preliminary exam. For purposes of the prelim exam, a question is considered substantially correct if you get 2 out of 3 subquestions essentially correct. To get a prelim **Ph.D. pass** you need to get at least 3 questions substantially correct. To get a prelim **masters pass** you need to get at least 2 questions substantially correct.

1. (a) A subset S of a group G is called a generating set if every element $x \in G$ can be written as a finite product of elements in S and their inverses. Let G be a connected topological group and U be an open neighborhood of the identity element $e \in G$. Show that U is a generating set for G. (Hint: Find a symmetric open neighborhood V of e within U and consider $\bigcup_{n=1}^{\infty} V^n$ where V^n is the set of all n-fold products of elements from V.)

(b) Let C be a countable subset of $\mathbb{R}^2$. Show that $\mathbb{R}^2 \setminus C$ is path connected.

(c) Let $GL_n(\mathbb{C})$ denote the complex general linear group topologized as a subspace of $Mat_n(\mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$. Show that $GL_n(\mathbb{C})$ is path connected.

3. (a) Recall that the Zariski topology on $\mathbb{R}^n$ is the topology where the closed sets are the common zero sets of families of real (multivariate) polynomials. Show that a proper Zariski closed set C of $\mathbb{R}^n$ must have empty interior in the standard Euclidean topology of $\mathbb{R}^n$ for all $n \geq 1$.

(b) Let $O(n) = \{A \in Mat_n(\mathbb{R}) | A^T A = I\}$ be the orthogonal group, topologized as a subspace of $Mat_n(\mathbb{R}) = \mathbb{R}^{n^2}$. Show that $O(n)$ is a compact smooth manifold. (Hint: Use the regular value theorem and the map $F : Mat_n(\mathbb{R}) \to S$. *Show that*

For questions 4 and 5, in addition to the standard basic facts of general topology, you may use all the following tools of differential topology freely as well as their basic properties: local immersion/submersion/inverse function theorems, regular value theorems, Sard's theorem, partitions of unity, tangent bundles, mod-2 intersection numbers, degrees and winding numbers.

4. (a) Let $f : M \to N$ be a smooth function between smooth, boundaryless manifolds and let Z be a submanifold of N. Suppose that for some point $z_0 \in Z$, there is an open neighborhood U of z_0 in N and a submersion $G : U \to \mathbb{R}^k$ such that $U \cap G^{-1}(0) = U \cap Z$. Finally suppose $f(x_0) = z_0$. **Clearly state the condition for f to be transverse to Z at x_0.** Then show that this condition is equivalent to the composite $G \circ f$ being a submersion at x_0. (You may use facts about the regular value theorem freely during this).

(b) Let p

5. (a) Define the concept of a **Morse function**. Then use Sard's theorem to show that given any smooth $g : U \to \mathbb{R}$ where U is an open subset of $\mathbb{R}^k$, that for almost every $a \in \mathbb{R}^k$, $g_a(x) = g(x) + a \cdot x$ is a Morse function.

(b) Let $f : H^{n-1} \to \mathbb{R}^n$ be a smooth map and let $z \in \mathbb{R}^n - f(H)$. State the conditions on H under which a mod-2 winding number $w_2(f, z)$ can be defined and write down an explicit map $u_{f,z} : H \to S^{n-1}$ for which $w_2(f, z) = deg_2(u_{f,z})$. Carefully state the **Jordan separation theorem** for smooth embeddings $f : H^{n-1} \to \mathbb{R}^n$ and briefly sketch how the mod-2 winding number can be used to show $\mathbb{R}^n - f(H)$ has at least 2 components. You do not have to provide an actual proof just the ideas and may quote the necessary properties of winding numbers in the sketch.

(c) State the conditions under which the mod-2 degree of a map $f : X \to Y$ can be defined. State the homotopy invariance and boundary map properties of this deg_2. Use deg_2 to prove that the following complex equation:

$$z^9 + \cos(|z|^2)(10z^8 + 3z^5 + \sin(|z|^2) + 3) = 0$$

has a solution $z \in \mathbb{C}$.