Topology Prelim Fall 2020.

March 9, 2023

There are 5 questions in this exam each consisting of 3 subquestions. The best 8 out of 15 subquestions will be used as your course nal score for MATH440. If you are a graduate student then this nal also counts as a

(b) Give an example of a space X and a subset A X with 2 A, the closure of A in X

codimension 1 submanifold of $\mathbb{R}^n$ for all $a > 0$. Finally prove that X_a is diffeomorphic to X_b for any $a, b > 0$.
(c) In this question you may use that metrizable spaces are normal/T_4 without proof. Use partitions of unity to prove the

Smooth Urysohn Lemma: Let M be a smooth manifold and A and B disjoint closed subsets in M. Then there exists a smooth function $F : M \to [0, 1]$ such that $F = 1$ on A and $F = 0$ on B.

5. (a) Let S^k denote the standard k-dimensional sphere (the set of unit vectors in $\mathbb{R}^{k+1}$). Prove, using Sard's theorem, that every smooth map $f : S^k \to S^n$ with $k < n$ is smoothly homotopic to a constant map.
(b) State the conditions of intersection theory i.e., the conditions on

$$f : X \to Y$$

and $Z \subset Y$ such that the mod-2 intersection number $I_2(f, Z)$ is defined. State the homotopy invariance property of these intersection numbers. Use this to find a map $f : S^1 \to T^2$ where T^2 is the 2-dimensional torus such that $f : S^1 \to T^2$ is not smoothly homotopic to a constant map. Then explain why this shows that S^2 is not diffeomorphic to T^2. (For this last part you may use the result of 5(a) freely even if you did not answer that subquestion. You may also describe f and compute intersection numbers $I_2(f, Z)$ pictorially though the rest of your answer should be more completely written out).
(c) State the conditions under which the mod-2 degree of a map $f : X \to Y$ can be defined. State the homotopy invariance and boundary map properties of this deg_2. Use deg_2 to prove that the following complex equation:

$$z^9 + \cos(|z|^2)(10z^8 + 3z^5 + \sin(|z|^2) + 3) = 0$$

has a solution $z \in \mathbb{C}$.