Real Analysis Preliminary Exam, August 2022

1. Let $(X; \mathcal{S}; \mu)$ be a measure space such that $\mu(X) < \infty$. Prove that if $\mathcal{A}$ is a family of disjoint sets in $\mathcal{S}$ such that $\mu(A) > 0$ for all $A \in \mathcal{A}$, then $\mathcal{A}$ is a countable set.

2. Let $(X; \mathcal{S}; \mu)$ be a measure space and consider $(f_n)_n \subset L^1(X)$ to be a sequence of functions converging pointwise a.e. to $f \in L^1(X)$. Show that

$$\lim_{n \to \infty} \int_X |f_n - f| \, d\mu = 0$$

if and only if

$$\lim_{n \to \infty} \int_X |f_n| \, d\mu = \int_X |f| \, d\mu .$$

3. If $h : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function, then its associated Hardy-Littlewood maximal function h